DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTERS OF SCIENCE-MATHEMATICS

SEMESTER -II

ORDINARY DIFFERENTIAL EQUATION

DEMATH2SCORE3

BLOCK-2

UNIVERSITY OF NORTH BENGAL

Postal Address:

The Registrar,

University of North Bengal,

Raja Rammohunpur,

P.O.-N.B.U.,Dist-Darjeeling,

West Bengal, Pin-734013,

India.

Phone: (O) +91 0353-2776331/2699008

Fax:(0353) 2776313, 2699001

Email: regnbu@sancharnet.in ; regnbu@nbu.ac.in

Wesbsite: www.nbu.ac.in

First Published in 2019

FOREWORD

The Self-Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university’s syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner’s interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

ORDINARY DIFFERENTIAL EQUATION

BLOCK-I

BLOCK-II

BLOCK-2 ORDINARY DIFFERENTIAL EQUATION

Introduction To Block

Unit 8 Higher order linear equation : deals with higher order linear equation and homogeneous linear differential equation

Unit 9 Wronskian and variation of constants : deals with wronskian and variation of constants also deals with wronskian theorem with its solution

Unit 10 Matrix exponential solutions : deals with matrix exponential solution and exponential differential equation and solution

Unit 11 Bvt for second order differential equation : deals with boundary value theorems for second order differential equation and solution

Unit 12 Green's function : deals with green's function, green's poison equation and fredholm theorems with its proof

Unit 13 Sturm comparison theorems and oscillations : deals with strum comparison theorem and oscillations with its proof

Unit 14 Eigenvalue problems : deals with eigenvalue problems and eigen oscillations, computation of eigen value, computation of eigen vectors and repeated eigen values

UNIT 8: HIGHER ORDER LINEAR EQUATIONS

STRUCTURE

8.0 OBJECTIVE

- In this unit we study higher order differential equation
- We also study Higher Order Linear Equations: Introduction and Basic Results

- We study Homogeneous Linear Equations With Constant Coefficients
- We study The Polar Form of a Complex Number
- We study Nonhomogeneous Linear Equations
- We study Equilibrium Points of Homogeneous Linear Systems

8.1 INTRODUCTION

In this chapter we're going to take a look at higher order differential equations. This chapter will actually contain more than most text books tend to have when they discuss higher order differential equations.

We will definitely cover the same material that most text books do here. However, in all the previous chapters all of our examples were 2nd order differential equations or 2×2 systems of differential equations. So, in this chapter we're also going to do a couple of examples here dealing with 3rd order or higher differential equations with Laplace transforms and series as well as a discussion of some larger systems of differential equations.

8.2 HIGHER ORDER DIFFERENTIAL EQUATIONS

Here is a brief listing of the topics in this chapter.

Basic Concepts for nth Order Linear Equations – In this section we'll start the chapter off with a quick look at some of the basic ideas behind solving higher order linear differential equations. Included will be updated definitions/facts for the Principle of Superposition, linearly independent functions and the Wronskian .

Linear Homogeneous Differential Equations – In this section we will extend the ideas behind solving 2nd order, linear, homogeneous differential equations to higher order. As we'll most of the process is identical with a few natural extensions to repeated real roots that occur more than twice. We will also need to discuss how to deal with repeated complex roots, which are now a possibility. In addition, we will see that the main difficulty in the higher order cases is simply finding all the roots of the characteristic polynomial.

Undetermined Coefficients – In this section we work a quick example to illustrate that using undetermined coefficients on higher order differential equations is no different that when we used it on 2nd order differential equations with only one small natural extension.

Variation of Parameters – In this section we will give a detailed discussion of the process for using variation of parameters for higher order differential equations. We will also develop a formula that can be used in these cases. We will also see that the work involved in using variation of parameters on higher order differential equations can be quite involved on occasion.

Laplace Transforms – In this section we will work a quick example using Laplace transforms to solve a differential equation on a 3rd order differential equation just to say that we looked at one with order higher than 2nd. As we'll see, outside of needing a formula for the Laplace transform of y''', which we can get from the general formula, there is no real difference in how Laplace transforms are used for higher order differential equations.

Systems of Differential Equations – In this section we'll take a quick look at extending the ideas we discussed for solving 2×2 systems of differential equations to systems of size 3×3. As we will see they are mostly just natural extensions of what we already know who to do. We will also make a couple of quick comments about 4×4 systems.

Series Solutions – In this section we are going to work a quick example illustrating that the process of finding series solutions for higher order differential equations is pretty much the same as that used on 2nd order differential equations.

8.3 HIGHER ORDER LINEAR EQUATIONS: INTRODUCTION AND BASIC RESULTS

Notes

Let us consider the equation

$$(NH) \qquad y^{(n)} + b_{n-1}(x)y^{(n-1)} + \ldots + b_1(x)y' + b_0(x)y = g(x)$$

,

and its associated homogeneous equation

$$(H) \qquad y^{(n)} + b_{n-1}(x)y^{(n-1)} + \ldots + b_1(x)y' + b_0(x)y = 0.$$

The following basic results hold:

(1)Superposition principle

Let $y_1, y_2, \ldots, y_k$ be solutions of the equation (*H*). Then, the function

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_k y_k$$

is also solution of the equation (*H*). This solution is called a linear combination of the functions $\{y_i\}$;

(2)The general solution of the equation (*H*) is given by

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n,$$

where $(c_1, c_2, \ldots, c_n)$ are arbitrary constants and $\{y_1, \ldots, y_n\}$ are *n* solutions of the equation (*H*) such that,

$$W(y_1, y_2, .., y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{n-1}(x) & y_2^{n-1}(x) & \cdots & y_n^{n-1}(x) \end{vmatrix} \neq 0.$$

In this case, we will say that $\{y_1, \ldots, y_n\}$ are linearly independent. The function $W(y_1, .., y_n)(x) = W(x)$ is called the Wronskian of $\{y_1, .., y_n\}$. We have

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} b_{n-1}(t)dt}.$$

Therefore, $W(x_0) \neq 0$, for some x_0, if and only if, $W(x) \neq 0$ for every *x*;

(3)The general solution of the equation (*NH*) is given by

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n + y_p,$$

where $(c_1, c_2, \ldots, c_n)$ are arbitrary constants, $\{y_1, \ldots, y_n\}$ are linearly independents solutions of the associated homogeneous equation (*H*), and $y_p(x)$ is a particular solution of (*NH*).

8.4 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Consider the *nth*-order linear equation with constant coefficients

$$(C) \qquad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0,$$

with $a_n \neq 0$. In order to generate *n* linearly independent solutions, we need to perform the following:

(1) Write the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0.$$

Then, look for the roots. These roots will be of two natures: simple or multiple. Let us show how they generate independent solutions of the equation(*H*).

(2)First case: Simple root

Let *r* be a simple root of the characteristic equation.

(2.1)

If *r* is a real number, then it generates the solution e^{rx};

(2.2)

Notes

If $r = \alpha + i\beta$ is a complex root, then since the coefficients of the characteristic equation are real, $\alpha - i\beta$ is also a root. The two roots generate the two solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$;

(3)Second case: Multiple root

Let *r* be a root of the characteristic equation with multiplicity *m*. If *r* is a real number, then generate the *m* independent solutions

$$e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx} \ .$$

If $r = \alpha + i\beta$ is a complex number, then $\alpha - i\beta$ is also a root with multiplicity *m*. The two complex roots will generate 2*m* independent solutions

$$\begin{cases} e^{\alpha x}\cos(\beta x), xe^{\alpha x}\cos(\beta x), \ldots, x^{m-1}e^{\alpha x}\cos(\beta x), \\ e^{\alpha x}\sin(\beta x), xe^{\alpha x}\sin(\beta x), \ldots, x^{m-1}e^{\alpha x}\sin(\beta x) \ . \end{cases}$$

Using properties of roots of polynomial equations, we will generate *n* independent solutions $\{y_1, \ldots, y_n\}$. Hence, the general solution of the equation (*H*) is given by

$$y = c_1 y_1 + \ldots + c_n y_n \ .$$

Therefore, the real problem in solving (*H*) has to do more with finding roots of polynomial equations. We urge students to practice on this.

Example: Find the general solution of

$$y^{(4)} + y = 0 \ .$$

Solution: Let us follow these steps:

(1) Characteristic equation

$$r^4 + 1 = 0 \ .$$

Its roots are the complex numbers

$$\cos\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{4}+k\frac{\pi}{2}\right).$$

In the analytical form, these roots are

$$\frac{1+i}{\sqrt{2}},\quad \frac{1-i}{\sqrt{2}},\quad \frac{-1+i}{\sqrt{2}},\quad \frac{-1-i}{\sqrt{2}}\ .;$$

(2) Independent set of solutions

(2.1) The complex roots $\frac{1+i}{\sqrt{2}}$ and $\frac{1-i}{\sqrt{2}}$ generate the two solutions

$$e^{x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right)\ \text{and}\ e^{x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right)\ .;$$

(2.2) The complex roots $\frac{-1+i}{\sqrt{2}}$ and $\frac{-1-i}{\sqrt{2}}$ generate the two solutions

$$e^{-x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right)\ \text{and}\ e^{-x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right)\ .;$$

(3) The general solution is

$$y=c_1e^{x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right)+c_2e^{x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right)+c_3e^{-x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right)+c_4e^{-x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right)\ .$$

As you may have noticed in this example, complex numbers do get involved very much in this kind of problem...

8.4.1 The Polar Form of a Complex Number

The fundamental trigonometric identity (i.e the Pythagorean theorem) is

$$\cos^2\theta+\sin^2\theta=1$$

From this we can see that the complex numbers

$$\cos\theta+i\sin\theta$$

Notes

are points on the circle of radius one centered at the origin.

Think of the point $\cos\theta + i\sin\theta$ moving counterclockwise around the circle as the real number θ moves from left to right. Similarly, the point moves clockwise if θ decreases. And whether θ increases or decreases, the point returns to the same position on the circle whenever θ changes by 2π or by 4π or by $2k\pi$ where k is any integer.

Exercise: Verify that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

Exercise: Prove *de Moivre's formula*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Now picture a fixed complex number on the unit circle

$$z = \cos\theta + i\sin\theta \qquad |z| = 1.$$

Consider multiples of z by a real, positive number r.

$$rz = r(\cos\theta + i\sin\theta) \qquad |rz| = r|z| = r.$$

As r grows from 1, our point moves out along the ray whose tail is at the origin and which passes through the point z. As r shrinks from 1 toward zero, our point moves inward along the same ray toward the origin. The modulus of the point is r. We call the angle θ which this ray makes with the x-axis, the *argument* of the number z. All the numbers rz have the same argument. We write

$$\arg rz = \theta$$

Just as a point in the plane is completely determined by its polar coordinates (r, θ), a complex number is completely determined by its modulus and its argument.

Notice that the argument is not defined when r=0 and in any case is only determined up to an integer multiple of 2π .

Why not just use polar coordinates? What's new about this way of thinking about points in the plane?

8.4.2 Non-homogeneous Linear Equations

Consider the nonhomogeneous linear equation

$$(NH) \qquad y^{(n)} + b_{n-1}(x)y^{(n-1)} + \ldots + b_1(x)y' + b_0(x)y = g(x)\ .$$

We have seen that the general solution is given by

$$y = y_p(x) + y_h,$$

where $y_p(x)$ is a particular solution and y_h is the general solution of the associated homogeneous equation. We will not discuss the case of non-constant coefficients. Therefore, we will restrict ourself only to the following type of equation:

$$(NH) \qquad a_n y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x)\ .$$

Using the previous section, we will discuss how to find the general solution of the associated homogeneous equation

$$(H) \qquad a_n y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0\ .$$

Therefore, the only remaining obstacle is to find a particular solution to (*NH*). In the second order differential equations case, we learned the two methods: Undetermined Coefficients Method and the Variation of Parameters. These two methods are still valid in the general case, but the second one is very hard to carry.

8.5.3 Method of Undetermined Coefficients or Guessing Method

As for the second order case, we have to satisfy two conditions. One is already satisfied since we assumed that our equation has constant

Notes

coefficients. The second condition has to do with the non-homogeneous term $g(x)$. Indeed, in order to use the undetermined coefficients method, $g(x)$ should be one of the elementary forms

$$g(x) = e^{\alpha x} P(x) \cos(\beta x) \quad \text{or} \quad g(x) = e^{\alpha x} P(x) \sin(\beta x),$$

where P_n is a polynomial function. For a more general case, see the remark below. In order to guess the form of the particular solution we follow these steps:

(1)Write down the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0 \ .$$

Find its roots and (especially) their multiplicity. Note that it will help strongly if you factorize this equation. This way you get the roots and their multiplicity;

(2)Write down the number $\alpha + i\beta$ (which you generate from g(x)). Then,

(2.1)if $\alpha + i\beta$ is not one of the root of the characteristic equation, then set s=0;

(2.2)if $\alpha + i\beta$ is one of the root of the characteristic equation, then s is its multiplicity;

(3)Write down the guessed form for the particular solution

$$y_p = x^s e^{\alpha x} \left(T(x) \cos(\beta x) + R(x) \sin(\beta x) \right),$$

where T(x) and R(x) are two polynomial functions with degree(T) = degree(R) = degree(P). So, if the degree of P is m, there are 2m+2 coefficients to be determined;

(4)Plug y_p into the equation (NH) to determine the coefficients of T and R;

(5)Write down your final answer for y_p .

Remark: The undetermined coefficients method can still be used if

$$g(x) = \sum_{i=1}^{m} g_i(x),$$

where $g_i(x)$ has the elementary form described above. Indeed, we need (as we did for the second order case) to split the equation (*NH*) into *m* equations. Find the particular solution to each one, then add them to generate the particular solution of the original equation.

Example: Find a particular solution of

$$y''' - 4y' = x + 3\cos(x)\ .$$

Solution: Let us follow these steps:

(1) Characteristic equation

$$r^3 - 4r = 0\ .$$

We have the factorization $r^3 - 4r = r(r-2)(r+2)$.

Therefore, the roots are 0,2,-2 and they are all simple.

(2) We have to split the equation into the following two equations:

$$\begin{array}{ll} (1) & y''' - 4y' = x \\ (2) & y''' - 4y' = 3\cos(x); \end{array}$$

(3) The particular solution to the equation (1):

(3.1) We have $\alpha + i\beta = 0$ which is a simple root. Then $s = 1$;

(3.2) The guessed form for the particular solution is

$$y_1 = x^1(Ax + B),$$

where *A* and *B* are to be determined. We will omit the detail of the calculations. We get *A* = -1/8 and *B*=0. Therefore, we have

$$y_1 = -\frac{1}{8}x^2 \ .;$$

(4) The particular solution to the equation (2):

(4.1) We have $\alpha + i\beta = i$ which is not a root. Then *s* = 0;

(4.2) The guessed form for the particular solution is

$$y_2 = x^0\left(A\cos(x) + B\sin(x)\right)$$

where *A* and *B* are to be determined. We will omit the detail of the calculations. We get *A* = 0 and *B*=-3/5. Therefore, we have

$$y_2 = -\frac{3}{5}\sin(x) \ .;$$

(5) The particular solution to the original equation is given by

$$y_p = -\frac{1}{8}x^2 - \frac{3}{5}\sin(x) \ .$$

Check In Progress-I

Q. 1 Find a particular solution of

$$y''' - 4y' = x + 3\cos(x) \ .$$

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

Q.2 Find the general solution of

$$y^{(4)} + y = 0 \ .$$

Solution : . . . .
. . . .
. . . .

. . . .

. . . .

. . . .

8.5 METHOD OF VARIATION OF PARAMETERS

This method is interesting whenever the previous method does not apply (when $g(x)$ is not of the desired form). The general idea is similar to what we did for second order linear equations except that, in that case, we were dealing with a small system and here we may be dealing with a bigger one (depending on the order of the differential equation). Let us describe the general case (constant coefficients or not). Consider the equation

$$(NH) \quad y^{(n)} + b_{n-1}(x)y^{(n-1)} + \ldots + b_1(x)y' + b_0(x)y = g(x).$$

Suppose that a set of independent solutions $\{y_1, \ldots, y_n\}$ of the associated homogeneous equation is known. Then a particular solution can be found as

$$y_p(x) = u_1(x)y_1 + u_2(x)y_2 + ... + u_n(x)y_n,$$

where the functions $u_i(x), i = 1, 2, ..$ can be obtained from the following system:

$$\begin{cases} u_1'(x)y_1 & + & u_2'(x)y_2 & + & \ldots & + & u_n'(x)y_n & = 0 \\ u_1'(x)y_1' & + & u_2'(x)y_2' & + & \ldots & + & u_n'(x)y_n' & = 0 \\ .... & & .... & & \ldots & & ... & \\ u_1'(x)y_1^{(n-1)} & + & u_2'(x)y_2^{(n-1)} & + & \ldots & + & u_n'(x)y_n^{(n-1)} & = g(x) \end{cases}$$

.

The determinant of this system is the Wronskian of $\{y_1, .., y_n\}$, which is not zero. Cramer's formulas will give

$$u_i'(x) = \frac{g(x)W_i(x)}{W(x)}, \quad i = 1, 2, .., n$$

,

Notes

where $W(x)$ is the Wronskian $W(y_1, .., y_n)$ and $W_i(x)$ is the determinant obtained from the Wronskian W by replacing the i^{th} - column in the vector column (0,0,..,0,1). Consequently, a particular solution to the equation (*NH*) is given by

$$y_p(x) = \sum_{i=1}^{i=n} y_i(x) \int \frac{g(x)W_i(x)}{W(x)} dx.$$

Note that when the order of the equation is not high, you may want to solve the system using techniques other than Cramer's formulas.

Example: Find a particular solution of

$$y''' + y' = \tan(x) \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Solution: Let us follow these steps:

(1) Characteristic equation

$$r^3 + r = 0 .$$

Since $r^3 + r = r(r^2 + 1)$, the roots of the characteristic equation are $0, \pm i$. Therefore, a set of independent solutions is $\{1, \cos(x), \sin(x)\}$;

(2)A particular solution is given by $y_p = u(x) + v(x)\cos(x) + w(x)\sin(x)$, where $\{u', v', w'\}$ are solutions of the system

$$\begin{cases} u' & +v'\cos(x) + w'\sin(x) & = & 0 \\ & -v'\sin(x) + w'\cos(x) & = & 0 \\ & -v'\cos(x) - w'\sin(x) & = & \tan(x) \end{cases};$$

(3) The resolution of the system gives

$$u' = \tan(x) \ , v' = -\cos(x)\tan(x), \ \text{ and } \ w' = -\sin(x)\tan(x) .$$

After integration we get

$$\begin{cases} u = \ln(\sec(x)) \\ v = \cos(x) \\ w = \sin(x) - \ln\Big(\sec(x) + \tan(x)\Big); \end{cases}$$

(4)A particular solution is given by

$$\begin{aligned} y_p &= \ln(\sec(x)) + \cos^2(x) + \sin^2(x) - \sin(x)\ln\Big(\sec(x) + \tan(x)\Big) \\ &= 1 + \ln(\sec(x)) - \sin(x)\ln\Big(\sec(x) + \tan(x)\Big). \end{aligned}$$

Note that the constant 1 in y_p may be dropped since it is the solution of the associated homogeneous equation.

8.6 LINEAR APPROXIMATIONS

This approximation is crucial to many known numerical techniques such as Euler's Method to approximate solutions to ordinary differential equations. The idea to use linear approximations rests in the closeness of the tangent line to the graph of the function around a point.

Let x_0 be in the domain of the function $f(x)$. The equation of the tangent line to the graph of $f(x)$ at the point (x_0,y_0), where $y_0 = f(x_0)$, is

$$y - y_0 = f'(x_0)(x - x_0).$$

If x_1 is close to x_0, we will write $x_1 = x_0 + \Delta x$, and we will approximate $f(x_0 + \Delta x)$ by the point (x_1,y_1) on the tangent line given by

$$y_1 = y_0 + \Delta x f'(x_0).$$

If we write $\Delta y = y_1 - y_0$, we have

$$\Delta y = \Delta x f'(x_0).$$

In fact, one way to remember this formula is to write $f'(x)$ as $\frac{dy}{dx}$ and then replace d by Δ. Recall that, when x is close to x_0, we have

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)\ .$$

Example. Estimate $\sqrt{9.2}$.

Let $f(x) = \sqrt{3+x}$. We have $f(6) = \sqrt{9} = 3$. Using the above approximation, we get

$$f(6.2) \approx f(6) + f'(6)(6.2 - 6)$$

We have

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x+3}}.$$

So $f'(6) = \frac{1}{6}$. Hence

$$f(6.2) \approx f(6) + f'(6)(6.2 - 6) = 3.033$$

or $\sqrt{9.2} \approx 3.033$. Check with your calculator and you'll see that this is a pretty good approximation for $\sqrt{9.2}$.

Remark. For a function $f(x)$, we define the **differential** df of $f(x)$ by

$$df = f'(x)\,dx \cdot$$

Example. Consider the function $y = f(x) = 5x^2$. Let Δx be an increment of x. Then, if Δy is the resulting increment of y, we have

$$\begin{aligned} \Delta y &= f(x + \Delta x) - f(x) \\ &= 5(x + \Delta x)^2 - 5x^2 \\ &= 10x(\Delta x) + 5(\Delta x)^2 \cdot \end{aligned}$$

On the other hand, we obtain for the differential dy:

$$dy = f'(x)\,dx = 10x\,dx\ .$$

In this example we are lucky in that we are able to compute Δy exactly, but in general this might be impossible. The error in the approximation, the difference between dy(replacing dx by Δx) and Δy, is $5(\Delta x)^2$, which is small compared to Δx.

Exercise 1. Use linear approximation to approximate

$$\sqrt[3]{65}.$$

Answer : We will use the fact that $\sqrt[3]{64} = 4$. Set

$$f(x) = \sqrt[3]{x} = x^{1/3}.$$

Then

$$f'(x) = \frac{1}{3}x^{(-2/3)}.$$

Thus

$$\sqrt[3]{65} = f(65) \approx f(64) + f'(64)\cdot(65-64) = 4 + \frac{1}{48} = 4.0208.$$

Exercise 2. Use linear approximation to approximate

$$\sin\left(\frac{\pi}{4}+0.02\right) .$$

Answer : Let us start by observing that if $f(x) = \sin x$, then both $f(\pi/4)$ and $f'(\pi/4)$ are known by elementary geometry:

$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2}; \quad f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2}.$$

Thus we can approximate

$$\sin\left(\frac{\pi}{4}+0.02\right) \approx f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\cdot 0.02 = \frac{1}{2}\sqrt{2} + \frac{0.02}{2}\sqrt{2} = 0.72125.$$

8.6.1 Linear Systems

Most of real life problems involve nonlinear systems (the predator-prey model is one such example). The nonlinear systems are very hard to solve explicitly, but qualitative and numerical techniques may help shed some information on the behavior of the solutions. But there are examples which are modeled by linear systems (the spring-mass model is one of them). Recall that a linear system of differential equations is given as

$$\begin{cases} \dfrac{dx}{dt} &= ax + by + f(t) \\ \dfrac{dy}{dt} &= cx + dy + g(t) \end{cases}$$

Example: The Harmonic Oscillator

This is a model for the motion of mass attached to a spring. Let *x*(*t*) be the displacement (which is the position of the mass from the equilibrium position or the rest position). Newton's Law of mechanics gives

$$m\frac{d^2x}{dt^2} = -k_s x - k_d\frac{dx}{dt} + F(t),$$

where $\frac{d^2x}{dt^2}$ is the acceleration, $-k_s x$ is the restoring force provided by the spring, $-k_d\frac{dx}{dt}$ is the damping force, and *F*(*t*) is an external force acting on the mass (such as an electrical field or a magnetic field for example). This is a second order differential equation. It may be translated to a system of first order differential system as

$$\begin{cases} \dfrac{dx}{dt} &= v \\ \dfrac{dv}{dt} &= -k_s x - k_d\dfrac{dx}{dt} + F(t) \end{cases}$$

where *v* is the velocity of the mass at time *t*. Clearly we have a linear system.

Definitions: If *f*(*t*)=*g*(*t*) = 0, then the linear system is called **homogeneous** and reduces to

$$\begin{cases} \dfrac{dx}{dt} &= ax + by \\ \dfrac{dy}{dt} &= cx + dy \end{cases}$$

Otherwise, it is called nonhomogeneous.

a, *b*, *c*, *d* are called the coefficients of the system. If all are constants then the system is said to be linear with constant coefficients.

Clearly, we may use our previous knowledge about systems for the linear ones. For example, we may associate the direction field to the linear system as well.

Example: Draw the direction field of the system

$$\begin{cases} \dfrac{dx}{dt} & = & -x + 4y \\ \dfrac{dy}{dt} & = & -3x - y \end{cases}$$

as well as some solutions.

8.6.2 Vector Representations of Solutions

Consider the linear system of differential equations

$$\begin{cases} \dfrac{dx}{dt} & = & ax + by + f(t) \\ \dfrac{dy}{dt} & = & cx + dy + g(t) \end{cases}$$

This system may be rewritten using matrix-notation. Indeed, set

$$Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

then the above system is equivalent to the matricial equation

$$\frac{dY}{dt} = \begin{pmatrix} ax + by + f(t) \\ cx + dy + g(t) \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

.

Using the matrix product, we get

$$\frac{dY}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} Y + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

.

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is called the coefficient matrix of the system. Note that the coefficients of the matrix *A* can be constant or not. The vector function

$$F(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

is called the nonhomogeneous term.

Remark: One may think that the equation above is only valid for linear systems of two equations. However, that is not the case. For example, consider the linear system

$$\begin{cases} \dfrac{dx}{dt} & = & -3x - 4y + \exp(t) \\[2ex] \dfrac{dy}{dt} & = & -y \\[2ex] \dfrac{dz}{dt} & = & -2z + 2x + \sin(t) \end{cases}$$

Then, in matricial notation, the system is equivalent to

$$\frac{dY}{dt} = \begin{pmatrix} -3 & -4 & 0 \\ 0 & -1 & 0 \\ 2 & 0 & -2 \end{pmatrix} Y + \begin{pmatrix} \exp(t) \\ 0 \\ \sin(t) \end{pmatrix},$$

where

$$Y(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

8.6.3 Equilibrium Points of Homogeneous Linear Systems

Consider the homogeneous linear system

$$\begin{cases} \dfrac{dx}{dt} &= ax + by \\ \dfrac{dy}{dt} &= cx + dy \end{cases}$$

The equilibrium points are given by the equations

$$\begin{cases} ax + by &= 0 \\ cx + dy &= 0 \end{cases}$$

Clearly, $x=0$ and $y=0$ give a trivial solution. Hence, the function $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives a constant solution to the linear system. We call it the trivial solution. In general, the equilibrium points are the intersection between two lines. Since the two lines intersect, they are the same (if parallel) or the intersection is reduced to one point. So, the set of equilibrium points is the entire line $ax+by=0$, or the trivial point (0,0). This conclusion is related to the determinant of the matrix coefficient. Indeed, if

$$det A = det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

is not equal to 0 (zero), then we have one equilibrium point (the trivial one).

Check In Progress-II

Q. 1 Define linear system.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

Q. 2 Find a particular solution of

$$y''' - 4y' = x + 3\cos(x)\ .$$

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

8.7 THE LINEARITY PRINCIPLE

This is may be the most important property for linear systems. Consider the homogeneous linear system

$$\frac{dY}{dt} = AY,$$

then

1. if *Y*(*t*) is a solution and *k* is a constant, then *k Y*(*t*) is also a solution;

2. if Y_1 and Y_2 are two solutions, then $Y_1 + Y_2$ is also a solution.

This clearly implies that if Y_1 and Y_2 are two solutions and k_1 and k_2 are two arbitrary constants, then

$$Y = k_1 Y_1 + k_2 Y_2$$

is also a solution. This conclusion is also known as the Principle of Superposition.

Clearly, from the Principle of Superposition, we may generate plenty of solutions once two solutions are known. The natural question to ask therefore, is whether we have obtained all the solutions. In order to better appreciate this problem let's consider the following example.

Example: Consider the linear system

$$\frac{dY}{dt} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} Y$$

Show that any solution *Y* to this system is given as

$$Y = k_1 Y_1 + k_2 Y_2,$$

where

$$Y_1 = \begin{pmatrix} e^t \\ 0 \end{pmatrix} \ , \ Y_2 = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

and k_1 and k_2 are two constants.

Answer: It is easy to check that indeed Y_1 and Y_2 are solutions to the given system. Let *Y* be any solution. Set

$$Y(0) = \begin{pmatrix} a \\ b \end{pmatrix}.$$

By the uniqueness and existence theorem, *Y* is the only solution to the IVP

$$\frac{dY}{dt} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} Y \ , \quad Y(0) = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Let us find k_1 and k_2 such that $Y = k_1 Y_1 + k_2 Y_2$. If this is the case, we should have $Y(0) = k_1 Y_1(0) + k_2 Y_2(0)$, which gives

$$\begin{pmatrix} a \\ b \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

which implies

$$\begin{cases} k_1 + k_2 = a \\ -2k_2 = b \end{cases}$$

Clearly, this gives

$$k_1 = a + \frac{1}{2}b \quad \text{and} \quad k_2 = -\frac{1}{2}b.$$

Consider the function

$$Y^* = \left(a + \frac{1}{2}b\right) Y_1 - \frac{1}{2}bY_2.$$

The linearity principle implies that Y^* is a solution. And, since

$$Y^*(0) = \begin{pmatrix} a \\ b \end{pmatrix},$$

the uniqueness and existence theorem implies that in fact $Y = Y^*$ gives the desired conclusion.

Remark: When you look at the above example you will notice that what made the conclusion work is that we were able to solve the algebraic system

$$\begin{pmatrix} a \\ b \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and this was possible because the two vectors

Notes

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

are linearly independent. In fact, the above conclusion is always valid whenever we have a linear independence around.

Theorem: The General Solution

Suppose Y_1 and Y_2 are two solutions to the linear system

$$\frac{dY}{dt} = AY.$$

Assume that the vectors $Y_1(0)$ and $Y_2(0)$ are linearly independent. Then, the solution to the IVP

$$\frac{dY}{dt} = AY , \quad Y(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

is given by

$$Y = k_1 Y_1 + k_2 Y_2,$$

for some constants k_1 and k_2. In this case, the two-parameter family

$$k_1 Y_1 + k_2 Y_2,$$

where k_1 and k_2 are arbitrary constants, is called the general solution of the system. Then, the two solutions Y_1 and Y_2 are said to be linearly independent.

Example: Consider the undammed harmonic oscillator

$$\frac{d^2 x}{dt^2} = -4x.$$

Show that any solution *x* is given by

$$x = k_1 \sin(2t) + k_2 \cos(2t).$$

Answer: Consider the associated linear system

$$\begin{cases} \dfrac{dx}{dt} & = & v \\ \dfrac{dv}{dt} & = & -4x \end{cases}$$

Set $Y = \begin{pmatrix} x \\ v \end{pmatrix}$. Note that the second component is just the derivative of the first one. Consider the two vector functions

$$Y_1 = \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} \text{ and } Y_2 = \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix}$$

It is easy to check that these two vector functions are in fact solutions to the given system. Also, you may check that the two vectors

$$Y_1(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \text{ and } Y_2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

are linearly independent. Therefore, any solution *Y* of the system is given by

$$Y = k_1 Y_1 + k_2 Y_2,$$

where k_1 and k_2 are two constants. Using the first component of *Y*, we see that any solution *x*(*t*) of the equation is given by

$$x = k_1 \sin(2t) + k_2 \cos(2t),$$

where k_1 and k_2 are two arbitrary constants.

8.8 LET'S SUM UP

- Here in this unit we study consider the equation

$$(NH) \qquad y^{(n)} + b_{n-1}(x)y^{(n-1)} + \ldots + b_1(x)y' + b_0(x)y = g(x)$$

,

and its associated homogeneous equation

$$(H) \qquad y^{(n)} + b_{n-1}(x)y^{(n-1)} + \ldots + b_1(x)y' + b_0(x)y = 0.$$

- We learnt *nth*-order linear equation with constant coefficients

$$(C) \qquad a_n y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0,$$

with $a_n \neq 0$. In order to generate *n* linearly independent solutions.

- We study characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0 .$$

- We study The equation of the tangent line to the graph of $f(x)$ at the point (x_0, y_0), where $y_0 = f(x_0)$, is

$$y - y_0 = f'(x_0)(x - x_0).$$

- We study the homogeneous linear system

$$\frac{dY}{dt} = AY$$

8.9 KEYWORD

Undamped : not damped or dampened; undiminished, as in energy, vigor

Harmonic Oscillator : A harmonic oscillator is a system that, when displaced from its equilibrium position, experiences a restoring force F proportional to the displacement x: ... If a frictional force (damping) proportional to the velocity is also present, the harmonic oscillator is described as a damped oscillator

Variation : A change or slight difference in condition, amount, or level, typically within certain limits

8.10 QUESTIONS FOR REVIEW

Q. 1 Consider the linear system

$$\frac{dY}{dt} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} Y$$

Show that any solution *Y* to this system is given as

$$Y = k_1 Y_1 + k_2 Y_2,$$

Where

$$Y_1 = \begin{pmatrix} e^t \\ 0 \end{pmatrix} \quad , \quad Y_2 = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

and k_1 and k_2 are two constants.

Q. 2 Use linear approximation to approximate

$$\sin\left(\frac{\pi}{4} + 0.02\right) .$$

Q. 3 Find the general solution of

$$y^{(4)} + y = 0 .$$

Q. 4 Find a particular solution of

$$y''' - 4y' = x + 3\cos(x) .$$

Q. 5 Use linear approximation to approximate

$$\sqrt[3]{65}.$$

8.11 SUGGESTION READING AND REFERENCES

- Halliday, David; Resnick, Robert (1977), Physics (3rd ed.), New York: Wiley, ISBN 0-471-71716-9
- Harper, Charlie (1976), Introduction to Mathematical Physics, New Jersey: Prentice-Hall, ISBN 0-13-487538-9
- Kreyszig, Erwin (1972), Advanced Engineering Mathematics (3rd ed.), New York: Wiley, ISBN 0-471-50728-8.
- Polyanin, A. D. and V. F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations (2nd edition)", Chapman & Hall/CRC Press, Boca Raton, 2003. ISBN 1-58488-297-2
- Simmons, George F. (1972), Differential Equations with Applications and Historical Notes, New York: McGraw-Hill, LCCN 75173716
- Tipler, Paul A. (1991), Physics for Scientists and Engineers: Extended version (3rd ed.), New York: Worth Publishers, ISBN 0-87901-432-6
- Boscain, Ugo; Chitour, Yacine (2011), Introduction à l'automatique (PDF) (in French)
- Dresner, Lawrence (1999), Applications of Lie's Theory of Ordinary and Partial Differential Equations, Bristol and Philadelphia: Institute of Physics Publishing, ISBN 978-0750305303

8.12 ANSWER TO CHECK IN PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 5.3

Q. 2 Check in Section 5

Check In progress-II

Answer Q. 1 Check in Section 7.1

Q. 2 Check in Section 6

UNIT 9: WRONSKIAN AND VARIATION OF CONSTANTS

STRUCTURE

9.0 OBJECTIVES

- In this unit we study Wronskian and Variation of Constants
- We also study Wronski Determinants
- We also study Linear Independence and the Wronskian
- WE STUDY LIOUVILLE-OSTROGRADSKI FORMULA
- We learn Variation of Constants with examples
- WE STUDY CAUCHY PROBLEM, CAUCHY PIANO WITH EXAMPLES
- We study Fundamental Solution of a linear partial differential equation

9.1 INTRODUCTION

The Wronskian of a set of n functions ϕ_1, ϕ_2, ... is defined by

$$W(\phi_1, \dots, \phi_n) \equiv \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}.$$

If the Wronskian is nonzero in some region, the functions ϕ_i are linearly independent. If $W = 0$ over some range, the functions are linearly dependent somewhere in the range.

The idea of the method of variation of constants is that the arbitrary constants participating in the general solution of the homogeneous system are replaced by functions of an independent variable. ... This formula is sometimes called the formula of variation of constants (cf. also Linear ordinary differential equation)

9.2 WRONSKIAN

Let y_1 and y_2 be two differentiable functions. We will say that y_1 and y_2 are proportional if and only if there exists a constant *C* such that $y_2 = Cy_1$. Clearly any function is proportional to the zero-function. If the constant *C* is not important in nature and we are only interested into the proportionality of the two functions, then we would like to come up with an equivalent criteria. The following statements are equivalent:

- y_1 and y_2 are proportional;
- $\frac{y_2}{y_1}$ is a constant function;
- $\left(\frac{y_2}{y_1}\right)' = 0$;
- $y_1y_2' - y_1'y_2 = 0$.

Therefore, we have the following:

y_1 and y_2 are not proportional if, and only if, $y_1y_2' - y_1'y_2 \neq 0$.

Define the Wronskian $W(y_1, y_2)$ of y_1 and y_2 to be $y_1 y_2' - y_1' y_2$, that is

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} .$$

The following formula is very useful

$$\left(\frac{y_2}{y_1}\right)' = \frac{W(y_1, y_2)}{y_1^2} .$$

Remark: Proportionality of two functions is equivalent to their linear dependence. Following the above discussion, we may use the Wronskian to determine the dependence or independence of two functions. In fact, the above discussion cannot be reproduced as is for more than two functions while the Wronskian does....

9.2.1 Wronski Determinant

The determinant of a system of n vector-functions of dimension n,

ϕi(t)={ϕ1i(t),…,ϕni(t)}, i=1,…,n

of the type

$$W(\phi_1(t), ..., \phi_n(t)) = \begin{vmatrix} \phi_1^1(t) & \dots & \phi_n^1(t) \\ \cdots & \dots & \dots \\ \phi_1^n(t) & \dots & \phi_n^n(t) \end{vmatrix} .$$

The Wronskian of a system of n scalar functions

$f_1(t), ..., f_n(t)$

which have derivatives up to order $(n-1)$ (inclusive) is the determinant

$$W(f_1(t), ..., f_n(t)) = \begin{vmatrix} f_1(t) & \dots & f_n(t) \\ f_1'(t) & \dots & f_n'(t) \\ \cdots & \dots & \dots \\ f_1^{(n-1)}(t) & \dots & f_n^{(n-1)}(t) \end{vmatrix} .$$

The concept was first introduced by J. Wronski [1].

Notes

If the vector-functions (1) are linearly dependent on a set E, then

$$W(\phi_1(t),\ldots,\phi_n(t)) \equiv 0, \qquad t \in E.$$

If the scalar functions (2) are linearly dependent on a set E, then

$$W(f_1(t),\ldots,f_n(t)) \equiv 0, \qquad t \in E.$$

The converse theorems are usually not true: Identical vanishing of a Wronskian on some set is not a sufficient condition for linear dependence of n functions on this set.

Let the vector-functions (1) be the solutions of a linear homogeneous n-th order system $\mathbf{x}' = A(t)\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^n$, with an $(n \times n)$-dimensional matrix $A(t)$ that is continuous on an interval I. If these solutions constitute a fundamental system, then

$$W(\phi_1(t),\ldots,\phi_n(t)) \neq 0, \qquad t \in I.$$

If the Wronskian of these solutions is equal to zero in at least one point of I, it is identically equal to zero on I, and the functions (1) are linearly dependent. The Liouville formula

$$W(\phi_1(t),\ldots,\phi_n(t)) =$$

$$= W(\phi_1(\tau),\ldots,\phi_n(\tau)) \exp \int_\tau^t \operatorname{Tr} A(s)\, ds, \qquad \tau, t \in I,$$

where $\operatorname{Tr} A(t)$ is the trace of the matrix $A(t)$, is applicable.

Let the functions (2) be the solutions of a linear homogeneous n-th order equation

$$y^{(n)} + p_1(t) y^{(n-1)} + \ldots + p_{n-1}(t) y' + p_n(t) y = 0$$

with continuous coefficients on the interval I. If these solutions constitute a fundamental system, then

$$W(f_1(t),\ldots,f_n(t)) \neq 0, \qquad t \in I.$$

If the Wronskian of these solutions is zero in at least one point of I, it is identically equal to zero on I, and the functions (2) are linearly dependent. The Liouville formula

$$W(f_1(t),\ldots,f_n(t)) =$$

$$= W(f_1(\tau),..., f_n(\tau))\exp\left[-\int_{\tau}^{t} p_1(s)\,ds\right], \qquad \tau, t \in I,$$

applies.

9.2.2 Linear Independence and the Wronskian

Let y_1 and y_2 be two differentiable functions. The **Wronskian** $W(y_1, y_2)$, associated to y_1 and y_2, is the function

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

For a discussion on the motivation behind the Wronskian,

We have the following important properties:

(1)*If* y_1 *and* y_2 *are two solutions of the equation y'' + p(x)y' + q(x)y = 0, then*

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)\exp\left(-\int_{x_0}^{x} p(t)dt\right).$$

(2)*If* y_1 *and* y_2 *are two solutions of the equation y'' + p(x)y' + q(x)y = 0, then*

$$W(y_1, y_2)(x) \neq 0 \text{ for every } x \iff \exists x_0 \text{ such that } W(y_1, y_2)(x_0) \neq 0.$$

In this case, we say that y_1 *and* y_2 *are linearly independent.*

(3)*If* y_1 *and* y_2 *are two linearly independent solutions of the equation y'' + p(x)y' + q(x)y = 0, then any solution y is given by*

$$y = c_1 y_1 + c_2 y_2$$

for some constant c_1 *and* c_2. *In this case, the set* $\{y_1, y_2\}$ *is called the* ***fundamental set*** *of solutions.*

Example: Let y_1 be the solution to the IVP

Notes

$$y'' + (2x - 1)y' + \sin(e^x)y = 0 \quad y(0) = 1, \quad y'(0) = -1 \; ;$$

and y_2 be the solution to the IVP

$$y'' + (2x - 1)y' + \sin(e^x)y = 0 \quad y(0) = 2, \quad y'(0) = 1 \; .$$

Find the Wronskian of $\{y_1, y_2\}$. Deduce the general solution to

$$y'' + (2x - 1)y' + \sin(e^x)y = 0 \; .$$

Solution: Let us write $W(x) = W(y_1, y_2)(x)$. We know from the properties that

$$W(x) = W(0)e^{-\int_0^x (2t-1)dt} = W(0)e^{-x^2 + x} \; .$$

Let us evaluate *W*(0). We have

$$W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1 + 2 = 3 \; .$$

Therefore, we have

$$W(x) = 3e^{-x^2+x} \; .$$

Since $W(0) \neq 0$, we deduce that $\{y_1, y_2\}$ is a fundamental set of solutions. Therefore, the general solution is given by

$$y = c_1 y_1 + c_2 y_2,$$

where c_1, c_2 are arbitrary constants.

9.3 LIOUVILLE-OSTROGRADSKI FORMULA

A relation that connects the Wronskian of a system of solutions and the coefficients of an ordinary linear differential equation.

Let $x_1(t),\dots,x_n(t)$ be an arbitrary system of n solutions of a homogeneous system of n linear first-order equations

$$\mathbf{x}' = A(t)\mathbf{x}, \qquad \mathbf{x}\in \mathbf{R}^n,$$

with an operator $A(t)$ that is continuous on an interval I, and let

$$W(x_1(t),\dots,x_n(t)) = W(t)$$

be the Wronskian of this system of solutions. The Liouville–Ostrogradski formula has the form

$$\frac{d}{dt}W(t) = W(t).\,\mathrm{Tr}\,A(t), \qquad t\in I,$$

or, equivalently,

$$W(x_1(t),\dots,x_n(t)) =$$

$$= W(x_1(t_0),\dots,x_n(t_0)).\exp\int_{t_0}^{t}\mathrm{Tr}\,A(s)\,ds, \qquad t,t_0\in I.$$

Here $\mathrm{Tr}\,A(t)$ is the trace of the operator $A(t)$. The Liouville–Ostrogradski formula can be written by means of the Cauchy operator $X(t,t_0)$ of the system (1) as follows:

$$\det X(t,t_0) = \exp\int_{t_0}^{t}\mathrm{Tr}\,A(s)\,ds, \qquad t,t_0\in I.$$

The geometrical meaning of (4) (or) is that as a result of the transformation $X(t,t_0):\mathbf{R}^n\rightarrow\mathbf{R}^n$ the oriented volume of any body is increased by a factor $\exp\int_{t_0}^{t}\mathrm{Tr}\,A(s)\,ds$.

If one considers a linear homogeneous n-th order equation

$$p_0(t)y^{(n)}+\dots+p_n(t)y = 0$$

with continuous coefficients on an interval I, and if $p_0(t)\neq 0$ for $t\in I$, then the Liouville–Ostrogradski formula is the equality

$$W(y_1(t),\dots,y_n(t)) =$$

$$= W(y_1(t_0),\dots,y_n(t_0)).\exp\left[-\int_{t_0}^{t}\frac{p_1(s)}{p_0(s)}\,ds\right], \qquad t, t_0 \in I,$$

where $W(y_1(t),\dots,y_n(t))$ is the Wronskian of the system of n solutions $y_1(t),\dots,y_n(t)$ of (5). The Liouville–Ostrogradski formulas , (6) are ordinarily used in the case when the system of solutions in question is fundamental (cf. Fundamental system of solutions). For example, formula (6) makes it possible to find by quadratures the general solution of a linear homogeneous equation of the second order if one knows one particular non-trivial solution of it.

The relation (6) for equation (5) with $n = 2$ was found by N.H. Abel in 1827 (see [1]), and for arbitrary n in 1838 by J. Liouville [2] and M.V. Ostrogradski [3]; the equality

was obtained by Liouville [2] and C.G.J. Jacobi [4] (as a consequence of this,

is sometimes called the Jacobi formula).

The Liouville–Ostrogradski formula (2) can be generalized to a non-linear system

$$x' = f(t, x), \qquad x \in \mathbf{R}^n,$$

under the assumption that the vector-valued function

$$f(t, x) = (f_1(t, x_1,\dots,x_n),\dots, f_n(t, x_1,\dots,x_n))$$

and the matrix $\partial f / \partial x$ are continuous. If $\Omega_{t_0} \subset \mathbf{R}^n$ is a set of finite measure $\mu(t_0)$ and the image Ω_t of this set under the linear mapping $X(t, t_0): \mathbf{R}^n \to \mathbf{R}^n$, where $X(t, t_0)$ is the Cauchy operator of the system (7), has measure $\mu(t)$, then

$$\frac{d\mu}{dt} = \int_{\Omega_t} \operatorname{div}_x f(t, x)\,dx;$$

here

$$\operatorname{div}_x f(t, x) = \sum_{i=1}^{n} \frac{\partial f_i(t, x_1,\dots,x_n)}{\partial x_i}.$$

This implies Liouville's theorem on the conservation of phase volume, which has important applications in the theory of dynamical systems and in statistical mechanics, mathematical problems in: The flow of a smooth autonomous system

$$x' = f(x), \qquad x \in \mathbf{R}^n,$$

does not change the volume of any body in the phase space $\mathbf{R}^n$ if and only if $\operatorname{div} f(x) = 0$ for all x; in particular, the phase volume is conserved by the flow of a Hamiltonian system.

4.1 Check In Progress-I

Q. 1 Define Wronskian.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

Q.2 Define Linear Independence of Wronskian.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

9.4 VARIATION OF CONSTANTS

A method for solving inhomogeneous (non-homogeneous) linear ordinary differential systems (or equations). For an inhomogeneous system, this method makes it possible to write down in closed form the general solution, if the general solution of the corresponding homogeneous system is known. The idea of the method of variation of constants is that the arbitrary constants participating in the general solution of the homogeneous system are replaced by functions of an independent variable. These functions must be chosen such that the inhomogeneous system is fulfilled. In concrete problems, this method was already applied by L. Euler and D. Bernoulli, but its complete elaboration was given by J.L. Lagrange .

Suppose one considers the Cauchy problem for the inhomogeneous linear system

$$\dot{x} = A(t)x + f(t), \qquad x(t_0) = x_0,$$

where

$$A: (\alpha, \beta) \to \text{Hom}\ (\mathbf{R}^n, \mathbf{R}^n),$$
$$f: (\alpha, \beta) \to \mathbf{R}^n$$

are mappings that are summable on every finite interval, and where $t_0 \in (\alpha, \beta)$. If $\Phi(t)$ is the fundamental matrix solution (cf. Fundamental solution) of the homogeneous system

$$\dot{y} = A(t)y,$$

then $y = \Phi(t)c$, $c \in \mathbf{R}^n$, is the general solution of (2). The method of variation of constants consists of a change of variable in (1):

$$x = \Phi(t)u,$$

and leads to the Cauchy formula for the solution of (1):

$$x = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(\tau)f(\tau)\,d\tau\ .$$

This formula is sometimes called the formula of variation of constants (cf. also Linear ordinary differential equation).

The idea of the method of variation of constants can sometimes be used in a more general non-linear situation for the description of the relation between the solution of a perturbed complete system and that of an unperturbed truncated system. E.g., for the solution $x(t)$ of the problem

$$\dot{x} = A(t)x + f(t, x), \qquad x(t_0) = x_0$$

(where A, f are continuous mappings and in the case of uniqueness of a solution) the formula of variation of constants is valid. It takes the form of the integral equation

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(\tau)f(\tau, x(\tau))\,d\tau .$$

In it, $\Phi(t)$ is the fundamental matrix solution of (2).

9.4.1 Cauchy Problem

One of the fundamental problems in the theory of (ordinary and partial) differential equations: To find a solution (an integral) of a differential equation satisfying what are known as initial conditions (initial data). The Cauchy problem usually appears in the analysis of processes defined by a differential law and an initial state, formulated mathematically in terms of a differential equation and an initial condition (hence the terminology and the choice of notation: The initial data are specified for $t = 0$ and the solution is required for $t \geq 0$). Cauchy problems differ from boundary value problems in that the domain in which the desired solution must be defined is not specified in advance. Nevertheless, Cauchy problems, like boundary value problems, are defined by the imposition of limiting conditions for the solution on (part of) the boundary of the domain of definition.

The main questions connected with Cauchy problems are as follows:

1) Does there exist (albeit only locally) a solution?

2) If the solution exists, to what space does it belong? In particular, what is its domain of existence?

3) Is the solution unique?

4) If the solution is unique, is the problem well-posed, i.e. is the solution in some sense a continuous function of the initial data?

The simplest Cauchy problem is to find a function $u(x)$ defined on the half-line $x \geq x_0$, satisfying a first-order ordinary differential equation

Notes

$$\frac{du}{dx} = f(x, u)$$

(f is a given function) and taking a specified value u_0 at $x = x_0$:

$$u(x_0) = u_0 .$$

In geometrical terms this means that, considering the family of integral curves of equation (1) in the (x, u)-plane, one wishes to find the curve passing through the point (x_0, u_0).

The first proposition concerning the existence of such a function (on the assumption that f is continuous for all x and continuously differentiable with respect to u) was proved by A.L. Cauchy (1820–1830) and generalized by E. Picard (1891–1896) (who replaced differentiability by a Lipschitz condition with respect to u). It turns out that under those conditions the Cauchy problem has a unique solution which, moreover, depends continuously on the initial data. Modern concepts of the Cauchy problem are essentially a far-reaching generalization of this problem.

The fact that questions 1) to 4) touch profoundly on the very heart of the matter — i.e. to answer them satisfactorily requires the imposition of certain conditions — is already illustrated in the theory of ordinary differential equations. Thus, a solution of the Cauchy problem for equation (1) with the condition (2), where f is given on an open set G and is only continuous, exists on some interval depending on G and (x_0, u_0) (see Peano theorem), but it need not be unique. The solution need not exist at all points in the domain of definition of f.

Repeating the above account almost word for word, one formulates the Cauchy problem for systems of ordinary differential equations, i.e. for an ordinary differential equation of type (1) with initial condition (2), where $u = u(x)$ is a function with values in a finite-dimensional vector space E, $u(x_0) = u_0 \in E$, and $f(x, u)$ is a function defined in $\mathbf{R}^+ \times E$. Here, again, the Picard conditions are sufficient for the existence and uniqueness of the solution and for the problem to be well-posed.

For ordinary differential equations of a higher order,

$$\frac{d^n u}{dx^n} = f(x, u, u', ..., u^{(n-1)}),$$

the Cauchy problem the initial data of which involve, besides the function itself, the derivatives

$$u(x_0)=u_0,\, u'(x_0)=u_0',\dots,u^{(n-1)}(x_0)=u_0^{(n-1)},$$

can be reduced by the standard device to a corresponding problem of type (1), (2).

In the case of first-order ordinary differential equations which cannot be expressed directly in terms of the derivative of the unknown function (as in equation (1)), the formulation of the Cauchy problem is similar, except that it relies to a high degree on the geometrical interpretation; however, the actual investigation of the equation may be complicated by the impossibility of (even locally) reducing the equation to the normal form (1).

While neither the formulation nor the investigation of the Cauchy problem for an ordinary differential equation involve essential difficulties, the situation is considerably more complicated in the case of partial differential equations (this applies, in particular, to answering questions 1) to 4)). This is true even if the functions involved are sufficiently regular (smooth). A major source of the difficulty is the fact that the space of independent variables is higher-dimensional, resulting in problems of (algebraic) solvability. E.g., consider the Cauchy problem for a system of equations in total differentials,

$$\omega^\alpha \overset{\text{def}}{=} \sum_i A_i^\alpha(x)\,dx^i=0,\qquad i=1,\dots,n,\qquad \alpha=1,\dots,k<n,$$

such equations being in a sense intermediate between "ordinary" and "partial" differential equations. The problem here is to determine an $(n-k)$-dimensional integral surface passing through a given point. Then the solvability condition is

$$d\,\omega^\alpha \bigwedge \omega^1 \bigwedge \dots \bigwedge \omega^k=0$$

(in a neighbourhood of the given point; here $d, \wedge$ are the symbols for the exterior differential and the exterior product, respectively) (see Frobenius theorem).

For linear partial differential equations

Notes

$$Lu = \sum_{|\alpha| \leq m} a_\alpha(x) \frac{\partial^\alpha u}{\partial x^\alpha} = f(x)$$

the Cauchy problem may be formulated as follows. In a certain region G of the variables $x = (x_1, \ldots, x_n)$ it is required to find a solution satisfying initial conditions, i.e. taking specified values, together with its derivatives of order up to and including $m-1$, on some $(n-1)$-dimensional hypersurface S in G. This hypersurface is known as the carrier of the initial conditions (or the initial surface). The initial conditions may be given in the form of derivatives of u with respect to the direction of the unit normal ν to S:

$$\left. \frac{\partial^k u}{\partial \nu^k} \right|_S = \phi_k, \qquad 0 \leq k \leq m-1,$$

where the $\phi_k(x)$, $x \in S$, are known functions (Cauchy data).

The formulation of the Cauchy problem for non-linear differential equations is similar.

A concept related to the Cauchy problem is that of a non-characteristic surface. If a non-singular coordinate transformation $x \to x'$ "straightens out" the surface S in a neighbourhood of x_0, i.e. it transforms it into a part of the hyperplane $x'_n = 0$, then the coefficient of $(\partial / \partial x'_n)^m$ in the transformed equation (3) is proportional to

$$Q(x, \nu) = \sum_{|\alpha| = m} a_\alpha(x) \nu^\alpha, \qquad \nu^\alpha = \nu_1^{\alpha_1} \ldots \nu_n^{\alpha_n}.$$

The surface S is said to be non-characteristic at the point x_0 if

$$Q(x_0, \nu) \neq 0.$$

In that case equation (3) may be written in a neighbourhood of x_0 in the so-called normal form:

$$\frac{\partial^m u}{\partial x_n'^m} = F\left(x'_i, \frac{\partial^\alpha u}{\partial x'^\alpha}\right), \qquad \alpha = (\alpha_1, \ldots, \alpha_n), \qquad \alpha_n < m.$$

Cauchy problems are usually studied when the carrier of the initial data is a non-characteristic surface, i.e. when condition (5) holds for all $x_0 \in S$.

The Cauchy–Kovalevskaya theorem occupies an important position in the theory of Cauchy problems; it runs as follows. If S is an analytic surface in a neighbourhood of one of its points x_0, if the functions a_α, f and ϕ_k, $0 \leq k \leq m-1$, are analytic in the same neighbourhood, and if moreover condition (5) is satisfied, then the Cauchy problem (3), (4) has an analytic solution $u(x)$ in a neighbourhood of the point; this solution is unique in the class of analytic functions. With the analyticity assumption, this theorem is also valid for general non-linear equations if the latter can be reduced to the normal form (6), and also for systems of such equations. The theorem is universal in nature, since it is applicable to analytic equations regardless of their type (elliptic, hyperbolic, etc.) and yields the local existence of a solution. The solution is unique in the class of non-analytic functions.

The Cauchy problem for partial differential equations of order exceeding 1 may turn out to be ill-posed if one drops the analyticity assumption for the equation or for the Cauchy data in the Cauchy–Kovalevskaya theorem. An illustration is Hadamard's example: The Cauchy problem for the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

with initial conditions

$$u(x, y, 0) = \phi_0 \ (x, y), \qquad \frac{\partial u}{\partial z}(x, y, 0) = 0$$

has no solution if $\phi_0 \ (x, y)$ is not an analytic function.

The hyperbolic equations constitute a broad class of equations for which the Cauchy problem is well-posed. In this case the Cauchy problem is global in nature, but the condition that S be non-characteristic is no longer sufficient. It is necessary that S is a space-like surface. A typical hyperbolic equation is the wave equation

$$\Box u = \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0,$$

considered in an $(n+1)$ -dimensional region, with variables $(x, t) = (x_1 ,..., x_n, t)$. The Cauchy problem for this equation with data

Notes

$$u(x,0) = \phi_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = \phi_1(x)$$

on the hyperplane $t=0$ is uniquely solvable for any sufficiently smooth functions ϕ_0, ϕ_1, and the solution depends continuously (in some C^k metric) on these functions. For the cases $n=1, 2$ and $n=3$, an explicit form of the solution is given by the formulas of d'Alembert, Poisson and Kirchhoff, respectively:

$$u(x,t) = \frac{1}{2}[\phi_0(x+t) + \phi_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \phi_1(\tau)\,d\tau;$$

$$u(x,t) = \frac{1}{2\pi}\int_{|y-x|^2 \le t^2} \frac{\phi_1(y)\,dy}{\sqrt{t^2 - |y-x|^2}} +$$

$$+\frac{1}{2\pi}\frac{\partial}{\partial t}\int_{|y-x|^2 \le t^2} \frac{\phi_0(y)\,dy}{\sqrt{t^2 - |y-x|^2}},$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$;

$$u(x,t) = \frac{1}{4\pi} t \int_{|\xi|=1} \phi_1(x + t\xi)\,d\sigma +$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial t}\left[t \int_{|\xi|=1} \phi_0(x + t\xi)\,d\sigma \right],$$

where $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$, and $d\sigma$ is the surface element on the unit sphere $|\xi| = 1$.

The set of points in the plane $t=0$ for which the Cauchy data completely determine the value $u(x,t)$ of the solution of the wave equation (7) at a point (x,t) is called the domain of dependence of the latter point. The domains of dependence of the point (x,t) in the cases $n=1, 2$ and $n=3$ are, respectively, the closed interval, disc and ball defined by $|y-x|^2 \le t^2$ (in the appropriate space $\mathbf{R}^n$). If the carrier of the Cauchy data is some region S on the hyperplane $t=0$, then the Cauchy data in that region affect the solution at all points (x,t) of the set such that the intersection $S \cap \{|y-x|^2 \le t^2\}$ is not empty; this set is known as the domain of influence.

The set of points $(x, t) \in \mathbf{R}^{n+1}$ at which the solution u is completely determined by the Cauchy data on S is called the domain of definition of $u(x, t)$ with initial data on S. In cases $n = 1, 2$ and 3 the domain of definition consists of all points (x, t) for which the closed interval, disc or ball, $|y - x|^2 \leq t^2$ (as the case may be), lies in S.

These results carry over to the more general case in which the carrier of the Cauchy data is a surface S of spatial type, i.e. a surface for which Q (see (5)) remains positive on S.

There are other problems besides the Cauchy problem which prove to be well-posed for hyperbolic equations; examples are the Cauchy characteristic problem and mixed initial-boundary value problems. In the latter type of problem, a solution exists in an $(n + 1)$-dimensional cylinder with generatrix parallel to the t-axis and a base S which is some region in the space of variables $x = (x_1, \ldots, x_n)$ with boundary Γ. The carrier of the initial conditions is S, while the value of the function, its normal derivative (in the case of second-order equations), or more general boundary value conditions, are given on the lateral surface $\Gamma \times \{t > 0\}$ of the cylinder.

In the case of degenerate equations the formulation of the Cauchy problem also has to be modified. For example, if the equation is of hyperbolic type and the carrier of the Cauchy data is a surface on which the equation becomes parabolically degenerate, then, depending on the nature of degeneracy, the initial value conditions may involve the use of some weight function.

9.5 FUNDAMENTAL SOLUTION

A solution of a partial differential equation $Lu(x) = 0$, $x \in \mathbf{R}^n$, with coefficients of class C^∞, in the form of a function $I(x, y)$ that satisfies, for fixed $y \in \mathbf{R}^n$, the equation

$$LI(x, y) = \delta(x - y), \qquad x \neq y,$$

which is interpreted in the sense of the theory of generalized functions, where δ is the delta-function. There is a fundamental solution for every partial differential equation with constant coefficients, and also for arbitrary elliptic equations. For example, for the elliptic equation

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0$$

with constant coefficients a_{ij} forming a positive-definite matrix α, a fundamental solution is provided by the function

$$I(x, y) = \begin{cases} \left[\sum_{i,j=1}^{n} A_{ij} (x_i - y_i)(x_j - y_j) \right]^{(2-n)/2}, & n > 2, \\ \log \left[\sum_{i,j=1}^{n} A_{ij} (x_i - y_i)(x_j - y_j) \right], & n = 2, \end{cases}$$

where A_{ij} is the cofactor of a_{ij} in the matrix α.

Fundamental solutions are widely used in the study of boundary value problems for elliptic equations.

Check In Progress-II

Q. 1 Define Variation of Constants.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

Q.2 Define Cauchy Problem.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

9.5.1 Fundamental System of Solutions

Fundamental system of solutions of a linear homogeneous system of ordinary differential equations

A basis of the vector space of real (complex) solutions of that system. (The system may also consist of a single equation.) In more detail, this definition can be formulated as follows.

A set of real (complex) solutions $\{x_1(t),\dots,x_n(t)\}$ (given on some set E) of a linear homogeneous system of ordinary differential equations is called a fundamental system of solutions of that system of equations (on E) if the following two conditions are both satisfied: 1) if the real (complex) numbers $C_1,\dots,C_n$ are such that the function

$$C_1 x_1(t)+\ldots+C_n x_n(t)$$

is identically zero on E, then all the numbers $C_1,\dots,C_n$ are zero; 2) for every real (complex) solution $x(t)$ of the system in question there are real (complex) numbers $C_1,\dots,C_n$ (not depending on t) such that

$$x(t)=C_1 x_1(t)+\ldots+C_n x_n(t) \quad \text{for all } t\in E.$$

If $(c_{ij})_{i,j=1}^{n}$ is an arbitrary non-singular $(n\times n)$-dimensional matrix, and $\{x_1(t),\dots,x_n(t)\}$ is a fundamental system of solutions, then $\{\sum_{j=1}^{n} c_{1j}x_j(t),\dots,\sum_{j=1}^{n} c_{nj}x_j(t)\}$ is also a fundamental system of solutions; every fundamental system of solutions can be obtained by such a transformation from a given one.

If a system of differential equations has the form

$$\dot{x}=A(t)x,$$

where $x\in\mathbf{R}^n$ (or $x\in\mathbf{C}^n$), if

$$A(.):(\alpha,\beta)\to \text{Hom}(\mathbf{R}^n,\mathbf{R}^n)$$
$$(\text{respectively } (\alpha,\beta)\to \text{Hom}(\mathbf{C}^n,\mathbf{C}^n))$$

and if the mapping $A(.)$ is summable on every segment contained in (α,β) ((α,β) is a bounded or unbounded interval in $\mathbf{R}$), then the vector space of solutions of this system is isomorphic to $\mathbf{R}^n$ (respectively, $\mathbf{C}^n$). Consequently, the system (1) has an infinite set of fundamental systems of solutions, and each such fundamental system consists of n solutions. For example, for the system of equations

$$\dot{u} = u, \qquad \dot{v} = -v,$$

an arbitrary fundamental system of solutions has the form

$$\left\{ \begin{pmatrix} e^t u_1 \\ e^{-t} v_1 \end{pmatrix}, \begin{pmatrix} e^t u_2 \\ e^{-t} v_2 \end{pmatrix} \right\},$$

where

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \qquad \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

are arbitrary linearly independent column vectors.

Every fundamental system of solutions of (1) has the form

$$\{X(t, \tau)x_1, \dots, X(t, \tau)x_n\},$$

where $X(t, \tau)$ is the Cauchy operator of (1), τ is an arbitrary fixed number in (α, β), and $x_1, \dots, x_n$ is an arbitrary fixed basis of $\mathbf{R}^n$ (respectively, $\mathbf{C}^n$).

If the system of differential equations consists of the single equation

$$x^{(k)} + a_1(t)x^{(k-1)} + \dots + a_k(t)x = 0,$$

where the functions

$$a_1(t), \dots, a_k(t) : (\alpha, \beta) \to \mathbf{R} \qquad (\text{or } (\alpha, \beta) \to \mathbf{C})$$

are summable on every segment contained in (α, β) ((α, β) is a bounded or unbounded interval in $\mathbf{R}$), then the vector space of solutions of this equation is isomorphic to $\mathbf{R}^k$ (respectively, $\mathbf{C}^k$). Consequently, the equation (2) has infinitely many fundamental sets of solutions, and each of them consists of k solutions. For example, the equation

$$\ddot{x} + \omega^2 x = 0, \qquad \omega \neq 0,$$

has fundamental system of solutions $\{\cos \omega t, \sin \omega t\}$; the general real solution of this equation is given by the formula

$$x = C_1 \cos \omega t + C_2 \sin \omega t,$$

where C_1 and C_2 are arbitrary real constants.

If a system of differential equations has the form

$$x^{(k)} = A_1(t)x^{(k-1)} + \ldots + A_k(t)x,$$

where $x \in \mathbf{R}^n$ (or $x \in \mathbf{C}^n$) and if for all $i = 1, \ldots, k-1$ the mappings

$$A_i(.): (\alpha, \beta) \to \text{Hom}(\mathbf{R}^n, \mathbf{R}^n)$$
$$(\text{or } (\alpha, \beta) \to \text{Hom}(\mathbf{C}^n, \mathbf{C}^n))$$

are summable on every segment contained in (α, β) (where (α, β) is a bounded or unbounded interval in $\mathbf{R}^n$), then the space of solutions of this system is isomorphic to $\mathbf{R}^{kn}$ (respectively, $\mathbf{C}^{kn}$); there are fundamental systems of solutions of (3), and each of them consists of kn solutions.

For linear homogeneous systems of differential equations that are not solved with respect to their leading derivatives, even if the coefficients of the system are constant, the number of solutions that appear in a fundamental system of solutions (that is, the dimension of the vector space of solutions) cannot always be calculated as easily as in the cases above. (In [1], Sect. 11 there is an examination of such a calculation for linear systems of differential equations with constant coefficients that are not solved with respect to their leading derivatives.

9.6 LET'S SUM UP

- We study Let y_1 and y_2 be two differentiable functions. We will say that y_1 and y_2 are proportional if and only if there exists a constant *C* such that $y_2 = Cy_1$
- We learnt system of n vector-functions of dimension n,
  ϕi(t)={ϕ1i(t),…,ϕni(t)}, i=1,…,n
- We learnt solution $x(t)$ of the variation of constant problem
  $$\dot{x} = A(t)x + f(t, x), \qquad x(t_0) = x_0$$
- We study The Liouville–Ostrogradski formula has the form
  $$\frac{d}{dt}W(t) = W(t). \text{Tr } A(t), \qquad t \in I,$$
- We study system of differential equations has the form
  $$x^{(k)} = A_1(t)x^{(k-1)} + \ldots + A_k(t)x,$$

Notes

where $x \in \mathbf{R}^n$ (or $x \in \mathbf{C}^n$) and if for all $i = 1, \ldots, k-1$ the mappings

9.7 KEYWORD

Operator : a person who operates equipment or a machine

Fundamental solution : a fundamental solution for a linear partial differential operator L is a formulation in the language of distribution theory of the older idea of a Green's function (although unlike Green's functions, fundamental solutions do not address boundary conditions).

Ostrogradski formula : A relation that connects the Wronskian of a system of solutions and the coefficients of an ordinary linear differential equation

9.8 QUESTIONS FOR REVIEW

Q. 1 Define Variation of Constants.

Q. 2 Define Linear Independence of Wronskian.

Q. 3 Define Wronski Determinant.

Q. 4 State Liouville-Ostrogradski Formula.

Q. 5 Write Eliptic Equation.

9.9 SUGGESTION READING AND REFERENCES

- T.M. Apostol, "Mathematical analysis" , Addison-Wesley (1974)
- P. Hartman, "Ordinary differential equations" , Birkhäuser (1982)
- G. Peano, "Sur le déterminant Wronskian" *Mathesis* , **9** (1889) pp. 75–76

- M. Böcher, "Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence" *Trans. Amer. Math. Soc.* , **2** (1901) pp. 139–149
- D.R. Curtis, "The vanishing of the Wronskian and the problem of linear dependence" *Math. Ann.* , **65** (1908) pp. 282–298
- K. Wolsson, "A condition equivalent to linear dependence for functions with vanishing Wronskian" *Linear Alg. Appl.* , **116** (1989) pp. 1–8
- K. Wolsson, "Linear dependence of a function set of m variables with vanishing generalized Wronskians" *Linear Alg. Appl.* , **117** (1989) pp. 73–80
- J.L. Lagrange, , Oeuvres , 4 , Paris (1869) pp. 151–251
- L.S. Pontryagin, "Ordinary differential equations" , Addison-Wesley (1962) (Translated from Russian)
- V.M. Alekseev, "An estimate for the perturbations of the solutions of ordinary differential equations" Vestnik Moskov. Univ. , 2 (1961) pp. 28–36 (In Russian) (English abstract)
- L.E. Reizin', "Local equivalence of differential equations" , Riga (1971) (In Russian)

9.10 ANSWER TO CHECK IN PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 3

Q. 2 Check in Section 3.2

Check In progress-II

Answer Q. 1 Check in Section 5

Q. 2 Check in Section 5.1

UNIT 10: MATRIX EXPONENTIAL SOLUTION

STRUCTURE

10.0 OBJECTIVE

- We study in this unit Matrix Exponential and its examples.
- WE ALSO STUDY EXPONENTIAL FUNCTION WITH ITS PROPERTIES
- WE STUDY HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS
- We study An equilibrium solution (or critical solution)
- We also learn Qualitative Analysis and Nullclines

10.1 INTRODUCTION

In mathematics, the matrix exponential is a matrix function on square matrices analogous to the ordinary exponential function. It is used to solve systems of linear differential equations. In the theory of Lie groups, the matrix exponential gives the connection between a matrix Lie algebra and the corresponding Lie group.

Consider a square matrix A of size n×n, elements of which may be either real or complex numbers. Since the matrix A is square, the operation of raising to a power is defined, i.e. we can calculate the matrices.

10.2 MATRIX EXPONENTIAL

The matrix exponential plays an important role in solving system of linear differential equations. On this page, we will define such an object and show its most important properties. The natural way of defining the exponential of a matrix is to go back to the exponential function e^x and find a definition which is easy to extend to matrices. Indeed, we know that the Taylor polynomials

$$T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

converges pointwise to e^x and uniformly whenever x is bounded. These algebraic polynomials may help us in defining the exponential of a matrix. Indeed, consider a square matrix A and define the sequence of matrices

$$A_n = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{n!}A^n.$$

When n gets large, this sequence of matrices get closer and closer to a certain matrix. This is not easy to show; it relies on the conclusion on e^x above. We write this limit matrix as e^A. This notation is natural due to the properties of this matrix. Thus we have the formula

$$e^A = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{n!}A^n + \cdots\cdots$$

One may also write this in series notation as

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!}A^n$$

At this point, the reader may feel a little lost about the definition above. To make this stuff clearer, let us discuss an easy case: diagonal matrices.

Example. Consider the diagonal matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to check that

$$A^n = \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}$$

for $n = 1, 2, \cdots$. Hence we have

$$I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{n!}A^n = \begin{pmatrix} 1 + \frac{2}{1!} + \cdots + \frac{2^n}{n!} & 0 \\ 0 & 1 + \frac{(-1)}{1!} + \cdots + \frac{(-1)^n}{n!} \end{pmatrix}.$$

Using the above properties of the exponential function, we deduce that

$$e^A = \begin{pmatrix} e^2 & 0 \\ 0 & e^{-1} \end{pmatrix}.$$

Indeed, for a diagonal matrix A, e^A can always be obtained by replacing the entries of A (on the diagonal) by their exponentials. Now let B be a matrix similar to A. As explained before, then there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

Moreover, we have

$$B^n = P^{-1}A^nP$$

for $n = 1, 2, \cdots$, which implies

$$I_n+\frac{1}{1!}B+\frac{1}{2!}B^2+\cdots+\frac{1}{n!}B^n = P^{-1}\left(I_n+\frac{1}{1!}A+\frac{1}{2!}A^2+\cdots+\frac{1}{n!}A^n\right)P.$$

This clearly implies that

$$e^B = P^{-1}\begin{pmatrix} e^2 & 0 \\ 0 & e^{-1} \end{pmatrix} P.$$

In fact, we have a more general conclusion. Indeed, let *A* and *B* be two square matrices. Assume that $A \sim B$. Then we have $e^A \sim e^B$. Moreover, if $B = P^{-1}AP$, then

$e^B = P^{-1}e^AP$.

Example. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This matrix is upper-triangular. Note that all the entries on the diagonal are 0. These types of matrices have a nice property. Let us discuss this for this example. First, note that

$$A^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathcal{O}.$$

In this case, we have

$$e^A = I + A + \frac{1}{2!}A^2 = \begin{pmatrix} 1 & 1 & 3/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, let *A* be a square upper-triangular matrix of order n. Assume that all its entries on the diagonal are equal to 0. Then we have

$$A^n = \mathcal{O}.$$

Such matrix is called a nilpotent matrix. In this case, we have

$$e^A = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{(n-1)!}A^{n-1}.$$

Notes

As we said before, the reasons for using the exponential notation for matrices reside in the following properties:

Theorem. The following properties hold:

1. $e^{O} = I_n$;
2. if A and B commute, meaning $AB = BA$, then we have
   $e^{A+B} = e^A e^B$;
3. for any matrix A, e^A is invertible and
   $\left(e^A\right)^{-1} = e^{-A}$.

The power series that defines the exponential map e^x also defines a map between matrices. In particular,

$$\exp(A) \equiv e^A \tag{1}$$

$$= \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{2}$$

$$= I + A + \frac{AA}{2!} + \frac{AAA}{3!} + \ldots, \tag{3}$$

converges for any square matrix A, where I is the identity matrix. The matrix exponential is implemented in the Wolfram Language as MatrixExp[*m*].

The Kronecker sum satisfies the nice property

$$\exp(A) \otimes \exp(B) = \exp(A \oplus B) \tag{4}$$

Matrix exponentials are important in the solution of systems of ordinary differential equations.

In some cases, it is a simple matter to express the matrix exponential. For example, when A is a diagonal matrix, exponentiation can be performed simply by exponentiating each of the diagonal elements. For example, given a diagonal matrix

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_k \end{bmatrix}, \qquad (5)$$

The matrix exponential is given by

$$\exp(A) = \begin{bmatrix} e^{a_1} & 0 & \cdots & 0 \\ 0 & e^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_k} \end{bmatrix}. \qquad (6)$$

Since most matrices are diagonalizable, it is easiest to diagonalize the matrix before exponentiating it.

When A is a nilpotent matrix, the exponential is given by a matrix polynomial because some power of A vanishes. For example, when

$$A = \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}, \qquad (7)$$

then

$$\exp(A) = \begin{bmatrix} 1 & x & z + \frac{1}{2} x y \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \qquad (8)$$

and $A^3 = 0$.

For the zero matrix $A = 0$,

$$e^0 = I, \qquad (9)$$

i.e., the identity matrix. In general,

$$e^A e^{-A} = e^0 = I, \qquad (10)$$

so the exponential of a matrix is always invertible, with inverse the exponential of the negative of the matrix. However, in general, the formula

$$e^A e^B = e^{A+B} \qquad (11)$$

holds only when A and B commute, i.e.,

$$[A, B] = AB - BA = 0. \tag{12}$$

For example,

$$\exp\left(\begin{bmatrix} 0 & -x \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}, \tag{13}$$

while

$$\exp\left(\begin{bmatrix} 0 & -x \\ 0 & 0 \end{bmatrix}\right)\exp\left(\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1-x^2 & -x \\ x & 1 \end{bmatrix}. \tag{14}$$

Even for a general 2×2 real matrix, however, the matrix exponential can be quite complicated

$$\exp\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{\Delta}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \tag{15}$$

where

$$m_{11} = e^{(a+d)/2}\left[\Delta \cosh\left(\tfrac{1}{2}\Delta\right) + (a-d)\sinh\left(\tfrac{1}{2}\Delta\right)\right] \tag{16}$$

$$m_{12} = 2\, b\, e^{(a+d)/2} \sinh\left(\tfrac{1}{2}\Delta\right) \tag{17}$$

$$m_{21} = 2\, c\, e^{(a+d)/2} \sinh\left(\tfrac{1}{2}\Delta\right) \tag{18}$$

$$m_{22} = e^{(a+d)/2}\left[\Delta \cosh\left(\tfrac{1}{2}\Delta\right) + (d-a)\sinh\left(\tfrac{1}{2}\Delta\right)\right], \tag{19}$$

and

$$\Delta \equiv \sqrt{(a-d)^2 + 4\, b\, c}\,. \tag{20}$$

As $\Delta \to 0$, this becomes

$$\exp\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = e^{(a+d)/2}\begin{bmatrix} 1 + \frac{1}{2}(a-d) & b \\ c & 1 - \frac{1}{2}(a-d) \end{bmatrix}. \tag{21}$$

10.3 MATRIX

A matrix is a concise and useful way of uniquely representing and working with linear transformations. In particular, every linear transformation can be represented by a matrix, and every matrix corresponds to a unique linear transformation. The matrix, and its close relative the determinant, are extremely important concepts in linear algebra, and were first formulated by Sylvester (1851) and Cayley.

In his 1851 paper, Sylvester wrote, "For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding of pth order." Because Sylvester was interested in the determinant formed from the rectangular array of number and not the array itself (Kline 1990, p. 804), Sylvester used the term "matrix" in its conventional usage to mean "the place from which something else originates" subsequently used the term matrix informally, stating "Form the rectangular matrix consisting of n rows and $(n+1)$ columns.... Then all the $n+1$ determinants that can be formed by rejecting any one column at pleasure out of this matrix are identically zero." However, it remained up to Sylvester's collaborator Cayley to use the terminology in its modern form in papers of 1855 and 1858.

In his 1867 treatise on determinants, C. L. Dodgson (Lewis Carroll) objected to the use of the term "matrix," stating, "I am aware that the word 'Matrix' is already in use to express the very meaning for which I use the word 'Block'; but surely the former word means rather the mould, or form, into which algebraical quantities may be introduced, than an actual assemblage of such quantities...." However, Dodgson's objections have passed unheeded and the term "matrix" has stuck.

The transformation given by the system of equations

$$x'_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \quad (1)$$

$$x'_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \quad (2)$$

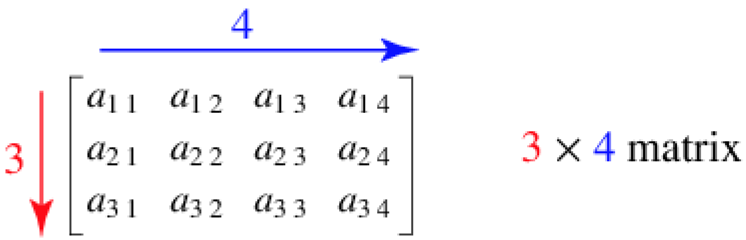
$$\vdots \quad (3)$$

$$x'_m = a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \quad (4)$$

is represented as a matrix equation by

$$\begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad (5)$$

where the a_{ij} are called matrix elements.



An $m \times n$ matrix consists of m rows and n columns, and the set of $m \times n$ matrices with real coefficients is sometimes denoted $\mathbb{R}^{m \times n}$. To remember which index refers to which direction, identify the indices of the last (i.e., lower right) term, so the indices m, n of the last element a_{34} in the above matrix identify it as an 3×4 matrix. Note that while this convention matches the one used for expressing measurements of a painting on canvas (where height comes first then width), it is opposite that used to measure paper, room dimensions, and windows, (in which the width is listed first followed by the height; e.g., 8 1/2 inch by 11 inch paper is 8 1/2 inches wide and 11 inches high).

A matrix is said to be square if $m = n$, and rectangular if $m \neq n$. An $m \times 1$ matrix is called a column vector, and a $1 \times n$ matrix is called a row vector. Special types of square matrices include the identity matrix I, with $a_{ij} = \delta_{ij}$ (where δ_{ij} is the Kronecker delta) and the diagonal matrix $a_{ij} = c_i \delta_{ij}$ (where c_i are a set of constants).

In this work, matrices are represented using square brackets as delimiters, but in the general literature, they are more commonly delimited using parentheses. This latter convention introduces the unfortunate notational ambiguity between matrices of the form $\begin{pmatrix} a \\ b \end{pmatrix}$ and the binomial coefficient

$$\binom{a}{b} = \frac{a!}{b!\,(a-b)!}. \quad (6)$$

When referenced symbolically in this work, matrices are denoted in a sans serif font, e.g, A, B, etc. In this concise notation, the transformation given in equation (5) can be written

$$\mathbf{x}' = \mathsf{A}\,\mathbf{x}, \tag{7}$$

where $\mathbf{x}'$ and $\mathbf{x}$ are vectors and A is a matrix. A number of other notational conventions also exist, with some authors preferring an italic typeface.

It is sometimes convenient to represent an entire matrix in terms of its matrix elements. Therefore, the (i, j)th element of the matrix A could be written a_{ij}, and the matrix composed of entries a_{ij} could be written as $\mathsf{A} = (a_{ij})_{ij}$, or simply $\mathsf{A} = (a)_{ij}$ for short.

Two matrices may be added (matrix addition) or multiplied (matrix multiplication) together to yield a new matrix. Other common operations on a single matrix are matrix diagonalization, matrix inversion, and transposition.

The determinant $\det(\mathsf{A})$ or $|\mathsf{A}|$ of a matrix A is a very important quantity which appears in many diverse applications. The sum of the diagonal elements of a square matrix is known as the matrix trace $\mathrm{Tr}(\mathsf{A})$ and is also an important quantity in many sorts of computations.

Check In Progress-I

Q. 1 Define Matrix Exponential.

Solution : . . . .
. . . .
. . . .
. . . .

Q.2 Define Matrix System of Equation.

Solution : . . . .
. . . .

10.4 EXPONENTIAL FUNCTION

The function

$$y = e^z \equiv \exp z,$$

where e is the base of the natural logarithm, which is also known as the Napier number. This function is defined for any value of z (real or complex) by

$$e^z = \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n,$$

and has the following properties:

$$e^{z_1} e^{z_2} = e^{z_1 + z_2} \quad \text{and} \quad (e^{z_1})^{z_2} = e^{z_1 z_2}$$

for any values of z_1 and z_2.

For real x, the graph of $y = e^x$ (the exponential curve) passes through the point $(0, 1)$ and tends asymptotically to the x-axis (see Fig.).

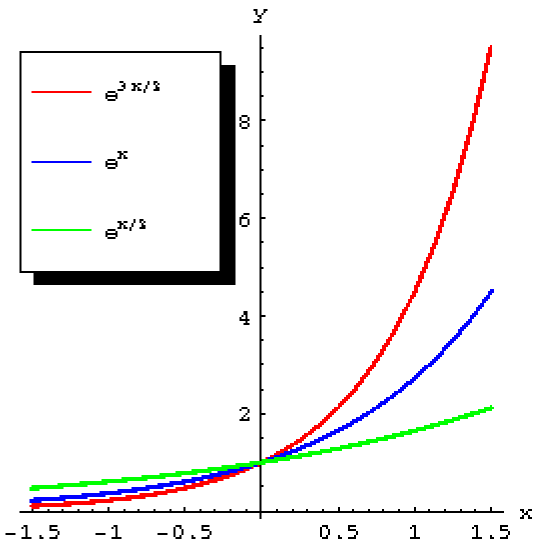


Figure: 5.1

In mathematical analysis one considers the exponential function $y = a^x$ for real x and $a > 0$, $a \neq 1$; this function is related to the (basic) exponential function $y = e^x$ by

$$a^x = e^{x \ln a}.$$

The exponential function $y = a^x$ is defined for all x and is positive, monotone (it increases if $a > 1$ and decreases if $0 < a < 1$), continuous, and infinitely differentiable; moreover,

$$(a^x)' = a^x \ln a, \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C,$$

and in particular

$$(e^x)' = e^x, \qquad \int e^x\,dx = e^x + C,$$

and in a neighbourhood of each point the exponential function can be expanded in a power series, for example:

$$e^x = 1 + \frac{x}{1!} + \ldots + \frac{x^n}{n!} + \ldots \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The graph of $y = a^x$ is symmetric about the ordinate axis to the graph of $y = (1/a)^x$. If $a > 1$, a^x increases more rapidly than any power of x as $x \to +\infty$, while as $x \to -\infty$ it tends to zero more rapidly than any power of $1/x$, i.e. for any natural number $b > 0$,

$$\lim_{x \to +\infty} \frac{a^x}{|x|^b} = \infty, \qquad \lim_{x \to -\infty} |x|^b a^x = 0.$$

The inverse of an exponential function is a logarithmic function.

If a and z are complex, the exponential function is related to the (basic) exponential function $w = e^z$ by

$$a^z = e^{z \operatorname{Ln} a},$$

where $\operatorname{Ln} a$ is the logarithm of the complex number a.

The exponential function $w = e^z$ is a transcendental function and is the analytic continuation of $y = e^x$ from the real axis into the complex plane.

An exponential function can be defined not only by (1) but also by means of the series (2), which converges throughout the complex plane, or by Euler's formula

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

Notes

If $z = x + iy$, then

$$|e^z| = e^x, \qquad \operatorname{Arg} e^z = y + 2\pi k, \qquad k = 0, \pm 1, \pm 2, \ldots .$$

The function e^z is periodic with period $2\pi i$: $e^{z+2\pi i} = e^z$. The function e^z assumes all complex values except zero; the equation $e^z = \alpha$ has an infinite number of solutions for any complex number $\alpha \neq 0$. These solutions are given by

$$z = \operatorname{Ln} \alpha = \ln |\alpha| + i \operatorname{Arg} \alpha .$$

The function e^z is one of the basic elementary functions. It is used to express, for example, the trigonometric and hyperbolic functions.

The basic exponential function $z \mapsto \exp(z)$ defined by (1) or, equivalently, (2) (with z instead of x) is single-valued. However, powers $z \mapsto \alpha^z$ for α complex $(\alpha \neq 0)$ are multiple-valued since $z \mapsto \operatorname{Ln} z$ denotes the "multiple-valued inverse" to $z \mapsto \exp(z)$. Thus, since it is customary to abbreviate $\exp(z)$ as e^z, the left-hand side of the identity

$$(e^{z_1})^{z_2} = e^{z_1 z_2}$$

is multiple-valued, while the right-hand side is single-valued. This identity is a dangerous one and should be dealt with with care, otherwise it may lead to nonsense like

$$1 = 1^{1/2} = (e^{2\pi i})^{1/2} = e^{\pi i} = -1 .$$

By considering a single-valued branch of the logarithm (cf. Branch of an analytic function), or by considering the complete analytic function Ln on its associated Riemann surface, an awkward notation and a lot of confusion may disappear. For fixed $\alpha \in \mathbf{C} \setminus 0$, any value (i.e. determination) of $\operatorname{Ln} \alpha$ defines an exponential function:

$$\alpha^z = e^{z(\text{value of } \operatorname{Ln}\alpha)} .$$

Problem. Let A and I_2 be the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Show that if $ad - bc \neq 0$ then A and I_2 are row equivalent. Recall that two matrices are row equivalent iff one may be obtained from the other one via row elementary operations.

Solution : Assume that $ad - bc \neq 0$. Let us show that A is row equivalent to I_2. Assume moreover that $a \neq 0$. Then divide the first row by a to get the new matrix

$$\begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix}$$

Now take the second row minus c times the first row to get

$$\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{cb}{a} \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad - bc}{a} \end{pmatrix}.$$

Divide the second row by $\frac{ad - bc}{a}$ since it not equal to 0 to get

$$\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

Finally take the first row minus $\frac{b}{a}$ times the first row to get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 .$$

Our proof is almost complete, if we show that the conclusion still holds when $a = 0$. In this case, neither b nor C are equal to 0. Switch the first row with the second one to get

$$\begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$

Divide the first row by c and the second row by b to get

$$\begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix}$$

Take the first row minus $\frac{d}{c}$ times the second row to get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \, .$$

The proof is now complete.

10.5 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Consider the *nth*-order linear equation with constant coefficients

$$(C) \qquad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0,$$

with $a_n \neq 0$. In order to generate *n* linearly independent solutions, we need to perform the following:

(1) Write the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0.$$

Then, look for the roots. These roots will be of two natures: simple or multiple. Let us show how they generate independent solutions of the equation(*H*).

(2)First case: Simple root

Let *r* be a simple root of the characteristic equation.

(2.1)

If *r* is a real number, then it generates the solution e^{rx} ;

(2.2)

If $r = \alpha + i\beta$ is a complex root, then since the coefficients of the characteristic equation are real, $\alpha - i\beta$ is also a root. The two roots generate the two solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$;

(3)Second case: Multiple root

Let *r* be a root of the characteristic equation with multiplicity *m*. If *r* is a real number, then generate the *m* independent solutions

$$e^{rx}, xe^{rx}, \ldots, x^{s-1}e^{rx} \; .$$

If $r = \alpha + i\beta$ is a complex number, then $\alpha - i\beta$ is also a root with multiplicity *m*. The two complex roots will generate 2*m* independent solutions

$$\begin{cases} e^{\alpha x}\cos(\beta x), xe^{\alpha x}\cos(\beta x), \ldots, x^{s-1}e^{\alpha x}\cos(\beta x), \\ e^{\alpha x}\sin(\beta x), xe^{\alpha x}\sin(\beta x), \ldots, x^{s-1}e^{\alpha x}\sin(\beta x) \; . \end{cases}$$

Using properties of roots of polynomial equations, we will generate *n* independent solutions $\{y_1, \ldots, y_n\}$. Hence, the general solution of the equation (*H*) is given by

$$y = c_1 y_1 + \ldots + c_n y_n \; .$$

Therefore, the real problem in solving (*H*) has to do more with finding roots of polynomial equations. We urge students to practice on this.

Example: Find the general solution of

$$y^{(4)} + y = 0 \; .$$

Solution: Let us follow these steps:

(1) Characteristic equation

Notes

$$r^4 + 1 = 0 .$$

Its roots are the complex numbers

$$\cos\left(\frac{\pi}{4} + k\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + k\frac{\pi}{2}\right) .$$

In the analytical form, these roots are

$$\frac{1+i}{\sqrt{2}}, \quad \frac{1-i}{\sqrt{2}}, \quad \frac{-1+i}{\sqrt{2}}, \quad \frac{-1-i}{\sqrt{2}} \; . ;$$

(2) Independent set of solutions

(2.1) The complex roots $\frac{1+i}{\sqrt{2}}$ and $\frac{1-i}{\sqrt{2}}$ generate the two solutions

$$e^{x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right) \quad \text{and} \quad e^{x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right) \; . ;$$

(2.2 The complex roots $\frac{-1+i}{\sqrt{2}}$ and $\frac{-1-i}{\sqrt{2}}$ generate the two solutions

$$e^{-x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right) \quad \text{and} \quad e^{-x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right) \; . ;$$

(3) The general solution is

$$y = c_1 e^{x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right) + c_2 e^{x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right) + c_3 e^{-x/\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}}\right) + c_4 e^{-x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}\right) .$$

As you may have noticed in this example, complex numbers do get involved very much in this kind of problem...

10.5.1 Introduction and Motivation

The differential equations are very much helpful in many areas of science. But most of interesting real life problems involve more than one unknown function. Therefore, the use of system of differential equations is very useful. Without loss of generality, we will concentrate on systems of two differential equations

$$\begin{cases} \dfrac{dx}{dt} = f(t,x,y) \\ \dfrac{dy}{dt} = g(t,x,y) \end{cases}$$

As a motivation let us consider an island with two type of species: Rabbits and Fox. Clearly one plays the role of predator while the other one the role of a prey. If we are interested to model the populations growths of both species, then we have to keep in mind that if, for example, the population of the Fox increases, then the Rabbit population will be affected. So the rate of change of the population of one type will depend on the actual population of the other type. For example, in the absence of the Rabbit population, the Fox population will decrease (and fast) to face a certain extinction. Something that most of us would like to avoid. A model for this Predator-Prey problem was developed by Lotka (in 1925) and Volterra (in 1926) and is known as the Lotka-Volterra system

$$\begin{cases} \dfrac{dR}{dt} = aR - \alpha RF \\ \dfrac{dF}{dt} = -bF + \beta RF \end{cases}$$

where $R(t)$ measures the Rabbit population, $F(t)$ measures the Fox population, and all the involved constant (a, b, α, γ) are positive numbers. Note that a and b are the growth rate of the prey, and the death rate of the predator. α and β are measures of the effect of the interaction between the Rabbits and The Fox.

Note that in the Lotka-Volterra system, the variable t is missing. This kind of system is called **autonomous system** and are written

Notes

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

10.5.2 Phase Plane

Let us go back to the general case

$$\begin{cases} \dfrac{dx}{dt} = f(t,x,y) \\ \dfrac{dy}{dt} = g(t,x,y) \end{cases}$$

A solution to this system is the couple of functions ($x(t)$,$y(t)$) which satisfy both differential equations of the system. When we change the variable t, then we get a set of points on the xy-plane which, in physics, we usually call a trajectory. The moving object has the coordinates ($x(t)$,$y(t)$) at time t. The velocity to the trajectory at time t is given by

$$\vec{V} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

Note that we do not need to know the solution ($x(t)$,$y(t)$ to determine the velocity vector at time t. Indeed, we have

$$\vec{V} = \left(f(x,y,t), g(x,y,t)\right)$$

as long as we know x, y, and t. In particular, we can draw all the velocity vectors everywhere on the plane for the autonomous systems. This is known as the vector field.

The vector field should be understood as the analogue of the direction field for differential equations.

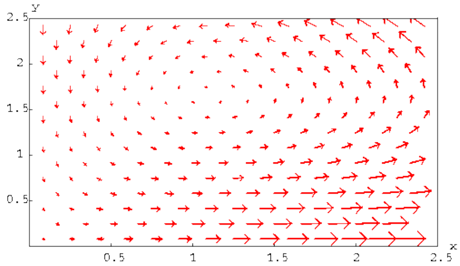
Example. Draw the vector field for the predator-prey problem

$$\begin{cases} \dfrac{dx}{dt} = 2x - 1.2xy \\ \dfrac{dy}{dt} = -y + 0.9xy \end{cases}$$

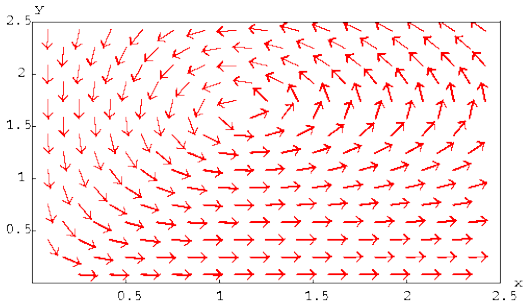
For example, for the point $(x,y) = (2,1)$, we have

$$\vec{V} = (1.6, 0.8)$$

If we put more velocity vectors, we get



It is clear that the length of the velocity vectors may affect our understanding of the solutions. If we are only interested by the direction of the motion not its speed, then it is natural to fix a length for these vectors

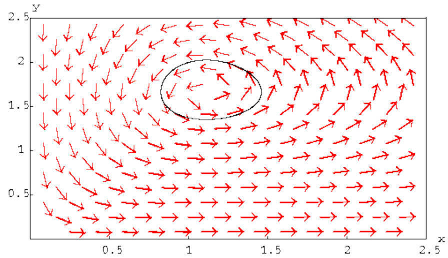


Compare the two pictures.

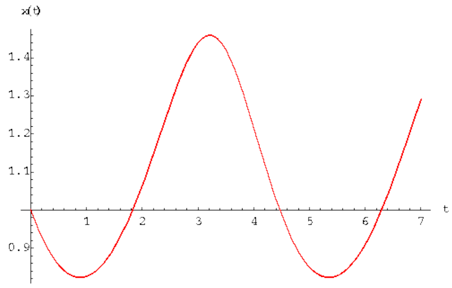
Remark. Note that for the predator-prey problem, we may be interested to find out whether one of the two species is facing extinction or not. In other words, we may be interested to study the functions $x(t)$ and $y(t)$ separately. In other words, two more graphs are naturally associated to a system. For the above example, the graph of the solution which satisfies the condition

$$\left(x(0), y(0)\right) = (1, 2)$$

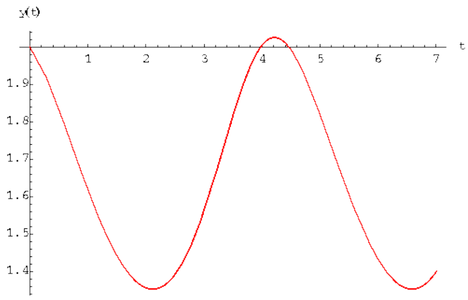
is drawn below



and the graphs of *x*(*t*) and *y*(*t*) are



and



10.5.3 Equilibrium Solutions

An **equilibrium solution** (or **critical solution**) of an autonomous system is the trajectory of a moving-object with no motion, that is the object is not moving. In this case, the velocity vector is basically equal to 0. Consider the autonomous system

$$\begin{cases} \dfrac{dx}{dt} &= f(x,y) \\ \\ \dfrac{dy}{dt} &= g(x,y) \end{cases}$$

The equilibrium solutions are the algebraic solutions of the system

$$\begin{cases} f(x,y) &= 0 \\ g(x,y) &= 0 \end{cases}$$

The equilibrium solutions are also called equilibrium points since the associated trajectories are exactly points on the phase plane.

Example. Find the equilibrium points of the Duffin system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \\ \dfrac{dy}{dt} &= -x + x^3 - y \end{cases}$$

Answer. We consider the system

$$\begin{cases} y &= 0 \\ -x + x^3 - y &= 0 \end{cases}$$

Plugging *y*=0 in the second equation, we get

$$-x + x^3 = 0$$

which gives *x* = 0, or $x^2 = 1$. Hence the equilibrium points are

$$(0,0),\ \ (1,0),\ \ \text{and}\ \ (-1,0)$$

Example: Consider the first order system

$$\begin{cases} x' &= -2x + 3y \\ y' &= -3x \end{cases}.$$

Is

Notes

$$\begin{cases} x(t) & = & e^{-t}\cos\left(2\sqrt{2}\,t\right) + 2\sqrt{2}e^{-t}\sin\left(2\sqrt{2}\,t\right) \\ y(t) & = & 3e^{-t}\cos\left(2\sqrt{2}\,t\right) \end{cases}$$

a solution of the system? Explain !

Answer: Clearly, we need to compute *x*' and *y*'. We have

$$\begin{cases} x'(t) & = & -e^{-t}\cos\left(2\sqrt{2}\,t\right) - 4\sqrt{2}e^{-t}\sin\left(2\sqrt{2}\,t\right) + 8e^{-t}\cos\left(2\sqrt{2}\,t\right) \\ y'(t) & = & -3e^{-t}\cos\left(2\sqrt{2}\,t\right) - 6\sqrt{2}e^{-t}\sin\left(2\sqrt{2}\,t\right) \end{cases}$$

.

Plug *x*' and *y*' into the system and check that they indeed satisfy both equations. Therefore, (*x*(*t*),*y*(*t*) is a solution.

Example: Write the second order differential equation

$$y'' + t^2 y' + 3y = 0$$

as a system of two first order differential equations.

Answer: set *y*'=*v*. Then we have

$$v' = y'' = -t^2 y' - 3y.$$

This yields

$$\begin{cases} y' & = & v \\ v' & = & -3y - t^2 v \end{cases}$$

Example: Find the solution to the system

$$\begin{cases} \dfrac{dx}{dt} & = & xy - x^2 - 3 \\ \\ \dfrac{dy}{dt} & = & y^2 - 4y + 4 \end{cases} \qquad Y_0 = (0, 2)\,.$$

Answer: First, solve the second equation since it does not contain the variable *x*. We recognize a separable equation. Hence, we will first look for the constant solutions.

$$y^2 - 4y + 4 = (y-2)^2 = 0.$$

This clearly gives y=2. The non-constant solutions can be obtained by separating the variables

$$\frac{dy}{(y-2)^2} = dt,$$

and then performing integration. Since

$$\int \frac{dy}{(y-2)^2} = -\frac{1}{y-2},$$

we get

$$-\frac{1}{y-2} = t + C\ .$$

If we put all the solutions together we get

$$\begin{cases} y & = & 2 \\ \dfrac{1}{y-2} & = & -t + C \end{cases}.$$

Clearly, the only solution satisfying the initial condition $y(0)=2$ is the constant solution y=2. Next, we plug the value of $y(t)$ into the first equation of the system to get

$$\frac{dx}{dt} = 2x - x^2 - 3\ .$$

This again is a separable equation. This time we do not have constant solutions since the quadratic equation $2x - x^2 - 3 = 0$ does not have real roots. Let us find the non-constant solutions. First, we separate the variables x and t to get

$$\frac{dx}{2x - x^2 - 3} = dt\ .$$

Notes

Since we have (using the techniques of integration of rational functions)

$$\int \frac{dx}{2x - x^2 - 3} = -\int \frac{dx}{x^2 - 2x + 3} = -\int \frac{dx}{(x-1)^2 + 2} = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)$$

,

then we get

$$-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) = t + C\ .$$

The initial condition $x(0)=0$ gives

$$C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\ .$$

Finally, the solution to the system is

$$-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) = t + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \text{and} \quad y = 2$$

.

You may want to find x explicitly as a function of t.

Remark: Since the constant solution $y=2$ is the solution of the second equation and the initial condition to be satisfied by y is $y(0)=2$, we may conclude directly from existence and uniqueness, that $y=2$ is the desired solution without looking for the non-constant solutions.

Example: Consider the following predator-prey model:

$$\begin{cases} x'(t) &= -x + 0.9xy \\ y'(t) &= 2y\left(1 - \frac{y}{2}\right) - 1.2xy \end{cases}$$

.

1. Does $x(t)$ denote the predator population or the prey population? Justify your answer.

2. Find all equilibrium points of the system.

3.Suppose the prey population becomes extinct while the predator population is still positive. Describe the long-term behavior of the predator population.

4.Suppose the predator population becomes extinct while the prey population is still positive. Describe the long-term behavior of the prey population.

5.Describe the long-term behavior of the system when the initial populations are given by

$$x(0) = \frac{20}{27} = 0.74 \quad \text{and} \quad y(0) = \frac{10}{9} = 1.11$$

.

Answer:

1.Recall that in the absence of prey, the population of predators decrease. It is clear that if y=0, then we have $x'(t)$ = -x, meaning that $x(t)$ will decrease. While, if we set x=0, we have y'=2y(1-y/2). Here we recognize the logistic equation which implies that y should get closer and closer to the carrying capacity y=2. Conclusion x represents the predator population.

2The equilibrium points are solutions of the system

$$\begin{cases} -x + 0.9xy & = \quad 0 \\ 2y\left(1 - \frac{y}{2}\right) - 1.2xy & = \quad 0 \end{cases}$$

.

Since,

$$-x + 0.9xy = 0 \quad \Leftrightarrow \quad x(-1 + 0.9y) = 0 \quad \Leftrightarrow \quad x = 0 \text{ or } y = \frac{1}{0.9} = \frac{10}{9}$$

,

we have the following two cases:

- **Case 1:** x=0, then the second equation gives

$$2y\left(1 - \frac{y}{2}\right) = 0 \quad \Leftrightarrow \quad y = 0 \text{ or } y = 2$$

.

Notes

Hence, we have two equilibrium points

$$(0,0) \text{ and } (0,2).$$

- **Case 2:** y=10/9, then the second equation gives

$$2\frac{10}{9}\left(1-\frac{10}{18}\right)-1.2x\frac{10}{9}=0,$$

which gives

$$x=\frac{20}{27}.$$

Hence, one equilibrium point (in this case)

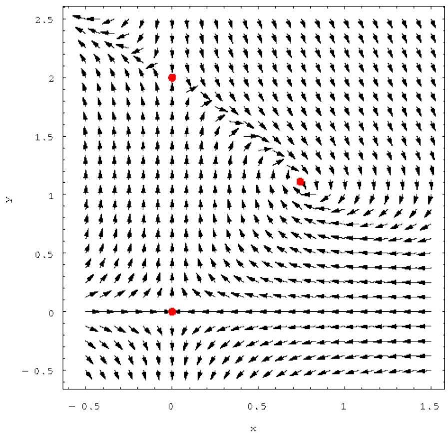
$$\left(\frac{20}{27},\frac{10}{9}\right).$$

Finally, the system has three equilibrium points

$$(0,0),\ (0,2) \text{ and } \left(\frac{20}{27},\frac{10}{9}\right).$$

3.It will become extinct.

4.It will approach the carrying capacity y=2.



5.Using the answer to 2, we see that the initial populations correspond to an equilibrium point. Therefore, both populations will remain unchanged

$$x=\frac{20}{27} \text{ and } y=\frac{10}{9}.$$

Example: Find the solution to the system

$$\begin{cases} \dfrac{dx}{dt} &= \tan(t)x + y + t \\ \dfrac{dy}{dt} &= \cos(t) \end{cases}$$

under the initial condition $Y_0 = (1, 0)$.

Answer: Notice that the second equation of the system is a differential equation involving only the variable *y*. Its integration gives

$$y(t) = \int \cos(t)dt = \sin(t) + C \ .$$

Note that $-\sin(t)$ is the derivative of $\cos(t)$ not its anti-derivative! The initial condition $Y_0 = (1, 0)$ translates into the initial condition *y*(0)=0 for the variable *y*. Hence, we have

$$y(0) = \sin(0) + C = 0 + C = C = 0,$$

which gives $y = \sin(t)$. Since we have *y*, we plug it into the first equation to get

$$\frac{dx}{dt} = \tan(t)x + \sin(t) + t \ .$$

We recognize a first order linear differential equation. In order to solve it, first we need to find the integrating factor given by

$$u(t) = e^{-\int \tan(t)dt} = e^{\ln|\cos(t)|} = \cos(t) \ .$$

Note that the anti-derivative of $\tan(t)$ is $-\ln|\cos(t)| = \ln|\sec(t)|$. The general solution is then given by

Notes

$$x(t) = \frac{\int \cos(t)\left(\sin(t)+t\right)}{\cos(t)}dt \ .$$

We have

$$\int \cos(t)\sin(t)dt = \frac{1}{2}\sin^2(t),$$

and

$$\int t\cos(t)dt = t\sin(t) - \int \sin(t)dt = t\sin(t) + \cos(t)$$

,

where, in the first integral, we used direct tables and for the second one we used integration by parts (we integrated $\cos(t)$ and differentiated *t*). Putting everything together, we get

$$x(t) = \frac{\frac{1}{2}\sin^2(t) + t\sin(t) + \cos(t) + C}{\cos(t)}.$$

The initial condition $Y_0 = (1,0)$ translates into the initial condition *x*(0)=1 for the variable *x*. Hence, we have

$$x(0) = 1 + C = 1$$

which gives *C*=0. Therefore, we have

$$x(t) = \frac{\frac{1}{2}\sin^2(t) + t\sin(t) + \cos(t)}{\cos(t)}.$$

Finally, the solution to the system is

$$\begin{cases} x(t) & = & \dfrac{\frac{1}{2}\sin^2(t) + t\sin(t) + \cos(t)}{\cos(t)} \\ y(t) & = & \sin(t) \end{cases} .$$

Note that since $\sin^2(t) = 1 - \cos^2(t)$, we may generate another expression for the function *x*(*t*).

Example: Consider the first order system

$$\begin{cases} x' & = & -2tx + 3y^2 \\ y' & = & -3x^2(1-y) \end{cases}$$

with the initial conditions

$$x(0) = -1 \text{ and } y(0) = 2 .$$

Use Euler's Method with step size h=0.1 to compute approximations for $x(t)$ and $y(t)$ at time t=0.1 and t=0.2.

Answer: We have

$$t_0 = 0 , \quad x_0 = -1 , \quad \text{and} \quad y_0 = 2 .$$

Using Euler's Method we know that

$$\begin{cases} t_1 & = & t_0 + h = 0.1 \\ x_1 & = & x_0 + x'(t_0)h = -1 + (-2t_0x_0 + 3y_0^2)0.1 = 0.2 \\ y_1 & = & y_0 + y'(t_0)h = 2 + \Big(-3x_0^2(1-y_0)\Big)0.1 = 2.3 \end{cases}$$

and

$$\begin{cases} t_2 & = & t_1 + h = 0.2 \\ x_2 & = & x_1 + x'(t_1)h \approx 1.783 \\ y_2 & = & y_1 + y'(t_1)h \approx 2.3156 \end{cases} .$$

Check In Progress-II

Q. 1 Find the equilibrium points of the Duffin system

Notes

$$\begin{cases} \dfrac{dx}{dt} & = & y \\ \\ \dfrac{dy}{dt} & = & -x + x^3 - y \end{cases}$$

Solution : . . . .

. . . .

. . . .

. . . .

Q.2 Find the solution to the system

$$\begin{cases} \dfrac{dx}{dt} & = & \tan(t)x + y + t \\ \\ \dfrac{dy}{dt} & = & \cos(t) \end{cases}$$

under the initial condition $Y_0 = (1, 0)$.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

10.6 QUALITATIVE ANALYSIS

Eery often it is almost impossible to find explicitly of implicitly the solutions of a system (specially nonlinear ones). The qualitative approach as well as numerical one are important since they allow us to make conclusions regardless whether we know or not the solutions.

Recall what we did for autonomous equations. First we looked for the equilibrium points and then, in conjunction with the existence and uniqueness theorem, we concluded that non-equilibrium solutions are either increasing or decreasing. This is the result of looking at the sign of the derivative. So what happened for autonomous systems? First recall that the components of the velocity vectors are $\frac{dx}{dt}$ and $\frac{dy}{dt}$. These vectors give the direction of the motion along the trajectories. We have the four natural directions (left-down, left-up, right-down, and right-up) and the other four directions (left, right, up, and down). These directions are obtained by looking at the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and whether they are equal to 0. If both are zero, then we have an equilibrium point.

Example. Consider the model describing two species competing for the same prey

$$\begin{cases} \dfrac{dx}{dt} & = & x(1-x) - xy \\ \\ \dfrac{dy}{dt} & = & 2y(1-y/2) - 3xy \end{cases}$$

Let us only focus on the first quadrant $x \geq 0$ and $y \geq 0$. First, we look for the equilibrium points. We must have

$$\begin{cases} x(1-x) - xy & = & 0 \\ 2y(1-y/2) - 3xy & = & 0 \end{cases}$$

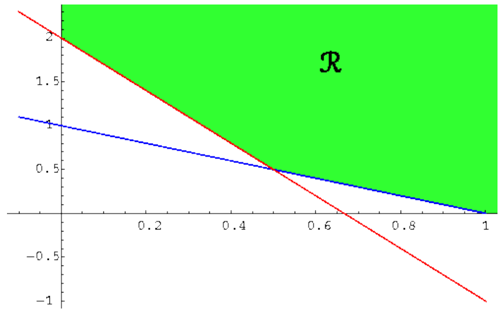
Algebraic manipulations imply

$$x = 0 \ \text{ or } \ 1 - x - y = 0$$

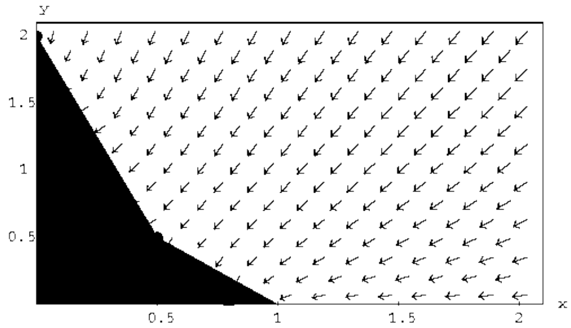
and

$$y = 0 \ \text{ or } \ 2 - 3x - y = 0$$

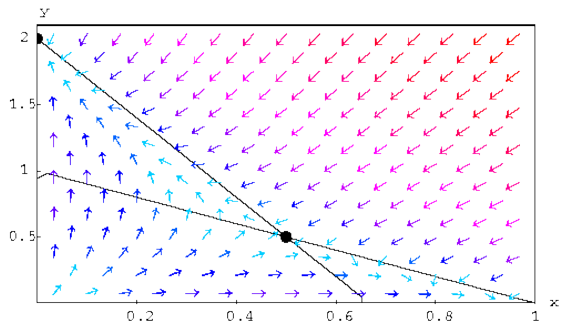
The equilibrium points are (0,0), (0,2), (1,0), and $\left(\frac{1}{2}, \frac{1}{2}\right)$. Consider the region **R** delimited by the x-axis, the y-axis, the line 1-*x*-*y*=0, and the line 2-3*x*-*y*=0.



Clearly inside this region neither $\frac{dx}{dt}$ or $\frac{dy}{dt}$ are equal to 0. Therefore, they must have constant sign (they are both negative). Hence the direction of the motion is the same (that is left-down) as long as the trajectory lives inside this region.



In fact, looking at the first-quadrant, we have three more regions to add to the above one. The direction of the motion depends on what region we are in (see the picture below)



The boundaries of these regions are very important in determining the direction of the motion along the trajectories. In fact, it helps to visualize the trajectories as slope-field did for autonomous equations. These boundaries are called **nullclines**.

Nullclines

Consider the autonomous system

$$\begin{cases} \dfrac{dx}{dt} & = & f(x,y) \\ \dfrac{dy}{dt} & = & g(x,y) \end{cases}$$

The **x-nullcline** is the set of points where $\frac{dx}{dt} = 0$ and **y-nullcline** is the set of points where $\frac{dy}{dt} = 0$. Clearly the points of intersection between x-nullcline and y-nullcline are exactly the equilibrium points. Note that along the x-nullcline the velocity vectors are vertical while along the y-nullcline the velocity vectors are horizontal. Note that as long as we are traveling along a nullcline without crossing an equilibrium point, then the direction of the velocity vector must be the same. Once we cross an equilibrium point, then we may have a change in the direction (from up to down, or right to left, and vice-versa).

Example. Draw the nullclines for the autonomous system and the velocity vectors along them.

$$\begin{cases} \dfrac{dx}{dt} & = & x(1-x) - xy \\ \dfrac{dy}{dt} & = & 2y(1-y/2) - 3xy \end{cases}$$

The x-nullcline are given by

$$\frac{dx}{dt} = x(1-x) - xy = 0$$

which is equivalent to

$$x = 0 \quad \text{or} \quad 1 - x - y = 0$$

Notes

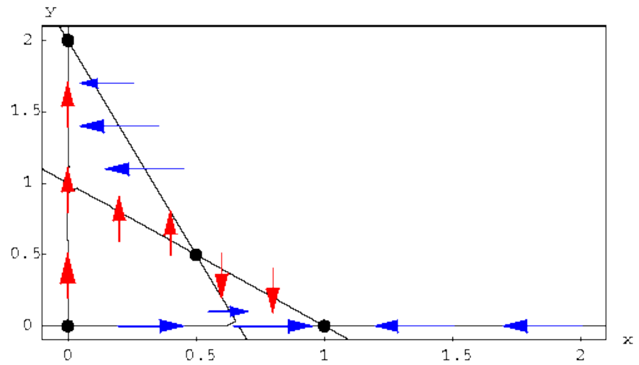
while the y-nullcline are given by

$$\frac{dy}{dt} = 2y(1 - y/2) - 3xy = 0$$

which is equivalent to

$$y = 0 \ \text{ or } \ 2 - 3x - y = 0$$

In order to find the direction of the velocity vectors along the nullclines, we pick a point on the nullcline and find the direction of the velocity vector at that point. The velocity vector along the segment of the nullcline delimited by equilibrium points which contains the given point will have the same direction. For example, consider the point (2,0). The velocity vector at this point is (-1,0). Therefore the velocity vector at any point (x,0), with $x > 1$, is horizontal (we are on the y-nullcline) and points to the left. The picture below gives the nullclines and the velocity vectors along them.



In this example, the nullclines are lines. In general we may have any kind of curves.

Example. Draw the nullclines for the autonomous system

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x) - xy \\[2ex] \dfrac{dy}{dt} = 2y(1 - y^2/2) - 3x^2y \end{cases}$$

The x-nullcline are given by

$$\frac{dx}{dt} = x(1-x) - xy = 0$$

which is equivalent to

$$x = 0 \text{ or } 1 - x - y = 0$$

while the y-nullcline are given by

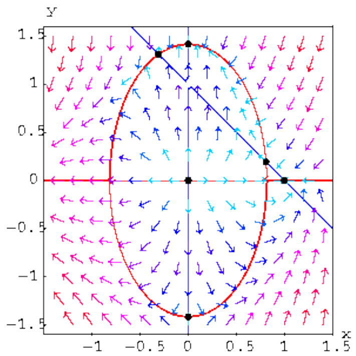
$$\frac{dy}{dt} = 2y(1 - y^2/2) - 3x^2y = 0$$

which is equivalent to

$$y = 0 \text{ or } 2 - 3x^2 - y^2 = 0$$

Hence the y-nullcline is the union of a line with the ellipse

$$3x^2 + y^2 = 2$$



10.6.1 Information from the Nullclines

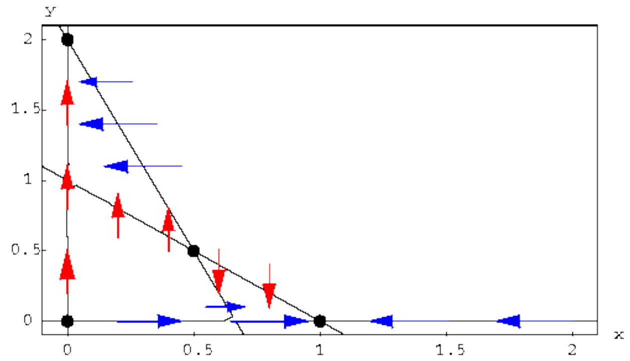
For most of the nonlinear autonomous systems, it is impossible to find explicitly the solutions. We may use numerical techniques to have an idea about the solutions, but qualitative analysis may be able to answer some questions with a low cost and faster than the numerical technique will do. For example, questions related to the long term behavior of solutions. The nullclines plays a central role in the qualitative approach. Let us illustrate this on the following example.

Example. Discuss the behavior of the solutions of the autonomous system

Notes

$$\begin{cases} \dfrac{dx}{dt} = x(1-x) - xy \\ \dfrac{dy}{dt} = 2y(1-y/2) - 3xy \end{cases}$$

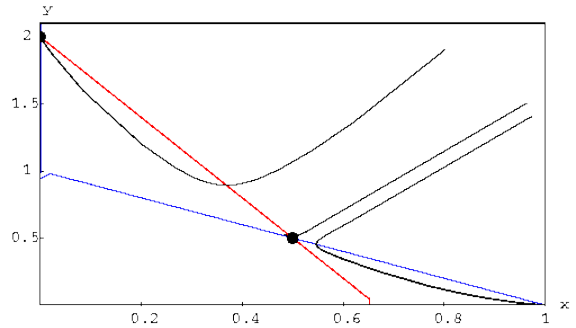
We have already found the nullclines and the direction of the velocity vectors along these nullclines.



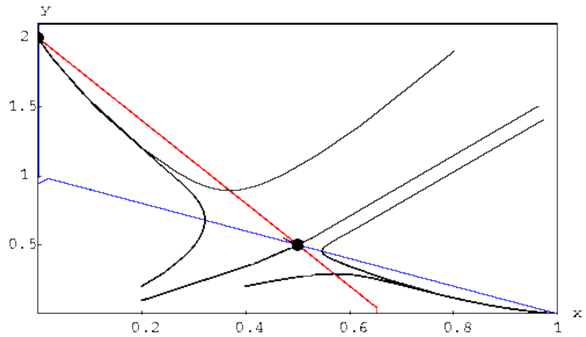
These nullclines give the birth to four regions in which the direction of the motion is constant. Let us discuss the region bordered by the x-axis, the y-axis, the line 1-*x*-*y*=0, and the line 2-3*x*-*y*=0. Then the direction of the motion is left-down. So a moving object starting at a position in this region, will follow a path going left-down. We have three choices

- **First choice:** the trajectory dies at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$.
- **Second choice:** the starting point is above the trajectory which dies at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Then the trajectory will hit the triangle defined by the points $\left(\frac{1}{2}, \frac{1}{2}\right)$, (0,1), and (0,2). Then it will go up-left and dies at the equilibrium point (0,2).
- **Third choice:** the starting point is below the trajectory which dies at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Then the trajectory will hit the triangle

defined by the points $\left(\frac{1}{2},\frac{1}{2}\right)$, (1,0), and $\left(\frac{2}{3},0\right)$. Then it will go down-right and dies at the equilibrium point (1,0).



For the other regions, look at the picture below. We included some solutions for every region.



Remarks. We see from this example that the trajectories which dye at the equilibrium point $\left(\frac{1}{2},\frac{1}{2}\right)$ are crucial to predicting the behavior of the solutions. These two trajectories are called **separatrix** because they separate the regions into different subregions with a specific behavior. To find them is a very difficult problem. Notice also that the equilibrium points (0,2) and (1,0) behave like sinks. The classification of equilibrium points will be discussed using the approximation by linear systems

10.7 LET'S SUM UP

We learnt in this unit that the Taylor polynomials

$$T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

We also learnt The function

$$y = e^{z} \equiv \exp z,$$

where e is the base of the natural logarithm, which is also known as the Napier number.

We study *nth*-order linear equation with constant coefficients

$$(C) \qquad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0,$$

with $a_n \neq 0$. In order to generate *n* linearly independent solutions.

In this section, we encountered the following important ideas:

- A slope field is a plot created by graphing the tangent lines of many different solutions to a differential equation.
- Once we have a slope field, we may sketch the graph of solutions by drawing a curve that is always tangent to the lines in the slope field.
- Autonomous differential equations sometimes have constant solutions that we call equilibrium solutions. These may be classified as stable or unstable, depending on the behavior of nearby solutions.

10.8 KEYWORD

Nullclines : The nullclines (null meaning zero, cline meaning slope) of the system x′=f(x,y),y′=g(x,y) occur when f(x,y)=0 or g(x,y)=0

Constant coefficients : the general second-order homogeneous linear differential equation has the form. If a(x), b(x), and c(x) are actually constants, a(x) ≡ a ≠ 0, b(x) ≡ b, c(x) ≡ c, then the equation becomes simply. This is the general second-order homogeneous linear equation with constant coefficients

Exponential : of or expressed by a mathematical exponent

10.9 QUESTIONS FOR REVIEW

Q. 1 Find the general solution of the system, using the matrix exponential:

dx/dt=2x+3y, dy/dt=3x+2y.

Q 2. Solve the system of equations by the method of matrix exponential:

Dx/dt=4x, dy/dt=x+4y.

Q 3 Solve the system of equations using the matrix exponential:

Dx/dt=x+y, dy/dt=−x+y.

Q. 4 Find the equilibrium points of the Duffin system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x + x^3 - y \end{cases}$$

Q. 5 Consider the following predator-prey model:

$$\begin{cases} x'(t) = -x + 0.9xy \\ y'(t) = 2y\left(1 - \frac{y}{2}\right) - 1.2xy \end{cases}.$$

1.Does *x*(*t*) denote the predator population or the prey population? Justify your answer.

2.Find all equilibrium points of the system.

3.Suppose the prey population becomes extinct while the predator population is still positive. Describe the long-term behavior of the predator population.

4.Suppose the predator population becomes extinct while the prey population is still positive. Describe the long-term behavior of the prey population.

5.Describe the long-term behavior of the system when the initial populations are given by

$$x(0) = \frac{20}{27} = 0.74 \quad \text{and} \quad y(0) = \frac{10}{9} = 1.11$$

10.10 SUGGESTION READING AND REFERENCES

K.R. Stromberg, "Introduction to classical real analysis" , Wadsworth (1981)

J.A. Dieudonné, "Foundations of modern analysis" , 1 , Acad. Press (1969) pp. 192 (Translated from French)

A.I. Markushevich, "Theory of functions of a complex variable" , 1 , Chelsea (1977) (Translated from Russian)

10.11 ANSWER TO CHECK IN PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 3

Q. 2 Check in Section 4

Check In progress-II

Answer Q. 1 Check in Section 6.3

Q. 2 Check in Section 6.

UNIT 11 : BOUNDARY VALUE PROBLEM FOR SECOND ORDER EQUATION

STRUCTURE

11.0 OBJECTIVES

- We study in this unit Boundary Value Problem for Second Order Differential Equation with solution
- We also study initial Value Problem with its examples.
- We learn three types of boundary conditions
  1. Dirichlet boundary conditions specify the value of the function on a surface $T = f(\mathbf{r}, t)$.
  2. Neumann boundary conditions specify the normal derivative of the function on a surface,

$$\frac{\partial T}{\partial n} = \hat{\mathbf{n}} \cdot \nabla T = f(\mathbf{r}, t).$$

3 Robin boundary conditions. For an elliptic partial differential equation in a region Ω, Robin boundary conditions specify the sum of αu and the normal derivative of $u = f$ at all points of the boundary of Ω, with α and f being prescribed.

- We also learn non-linear boundary value problems

11.1 INTRODUCTION

In mathematics, in the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

Boundary value problems arise in several branches of physics as any physical differential equation will have them. Problems involving the wave equation, such as the determination of normal modes, are often stated as boundary value problems. A large class of important boundary value problems are the Sturm–Liouville problems. The analysis of these problems involves the eigen functions of a differential operator.

To be useful in applications, a boundary value problem should be well posed. This means that given the input to the problem there exists a unique solution, which depends continuously on the input. Much theoretical work in the field of partial differential equations is devoted to proving that boundary value problems arising from scientific and engineering applications are in fact well-posed.

Among the earliest boundary value problems to be studied is the Dirichlet problem, of finding the harmonic functions (solutions to Laplace's equation); the solution was given by the Dirichlet's principle.

A boundary value problem (BVP) for an ordinary differential equation (ODE) will consist of an ODE together with conditions specified at more than one point. In particular, we will be con cer n ed with solving scalar differential equations,

$y(n)= f(x, y, y', \ldots, y(n-1)), n \geq 2,$

where f is real- valued and boundary conditions (BC's) on solutions of the equation are
specified at k, (with $k \geq 2$), points belonging to some interval of the reals .Let us first consider some difficulties which might occur. A two-point boundary value problem (BVP) of total order *n* on a finite interval $[a,b]$ may be written as an explicit first order system of ordinary differential equations (ODEs) with boundary values evaluated at two points as

$$y'(x) = f(x,y(x)), \quad x \in (a,b), \quad g(y(a)), \quad y(b)) = 0 \qquad (1)$$

Here, $y,f,g \in Rn$ and the system is called explicit because the derivative y' appears explicitly. The *n* boundary conditions defined by *g* must be independent; that is, they cannot be expressed in terms of each other (if *g* is linear the boundary conditions must be linearly independent).

In practice, most BVPs do not arise directly in the form (1) but instead as a combination of equations defining various orders of derivatives of the variables which sum to *n* . In an explicit BVP system, the boundary conditions and the right hand sides of the ordinary differential equations (ODEs) can involve the derivatives of each solution variable up to an order one less than the highest derivative of that variable appearing on the left hand side of the ODE defining the variable. To write a general system of ODEs of different orders in the form (1), we can define *y* as a vector made up of all the solution variables and their derivatives up to one less than the highest derivative of each variable, then add trivial ODEs to define these derivatives. See the section on initial value problems for an example of how this is achieved. See also Ascher et al.(1995) who show techniques for rewriting boundary value problems of various orders as first order systems. Such rewritten systems may not be unique and do not necessarily provide the most efficient approach for computational solution.

The words *two-point* refer to the fact that the boundary condition function *g* is evaluated at the solution at the two interval endpoints *a* and *b* unlike for initial value problems (IVPs) where the *n* initial conditions are all evaluated at a single point. Occasionally, problems arise where the function g is also evaluated at the solution at other points in (a,b). In these cases, we have a multipoint BVP. As shown in Ascher et al. (1995), a multipoint problem may be converted to a two-point problem by defining separate sets of variables for each subinterval between the points and adding boundary conditions which ensure continuity of the variables across the whole interval. Like rewriting the original BVP in the compact form (1), rewriting a multipoint problem as a two-point problem may not lead to a problem with the most efficient computational solution.

Most practically arising two-point BVPs have separated boundary conditions where the function g may be split into two parts (one for each endpoint):

$Ga\ (y(a))=0, \quad gb(y(b)\ ) = 0.$

Here, $ga \in Rs$ and $gb \in Rn-s$ for some value s with $1<s<n$ and where each of the vector functions ga and gb are independent. However, there are well-known, commonly arising, boundary conditions which are not separated; for example, consider periodic boundary conditions which, for a problem written in the form of equation (1), are

$y(a) - y(b) = 0.$

11.2 EXPLANATION

Boundary value problems are similar to initial value problems. A boundary value problem has conditions specified at the extremes ("boundaries") of the independent variable in the equation whereas an initial value problem has all of the conditions specified at the same value of the independent variable (and that value is at the lower boundary of the domain, thus the term "initial" value). A boundary value is a data value that corresponds to a minimum or maximum input, internal, or output value specified for a system or component.

For example, if the independent variable is time over the domain [0,1], a boundary value problem would specify values for y(t) at both t = 0 and t = 1, whereas an initial value problem would specify a value of y(t) and y’(t) at time t = 0.

Finding the temperature at all points of an iron bar with one end kept at absolute zero and the other end at the freezing point of water would be a boundary value problem.

If the problem is dependent on both space and time, one could specify the value of the problem at a given point for all time or at a given time for all space.

Example 1.1. Linear equations (Initial Value Problems (IVP's) have

11.3 BOUNDARY VALUE PROBLEM

A boundary value problem is a problem, typically an ordinary differential equation or a partial differential equation, which has values assigned on the physical boundary of the domain in which the problem is specified. For example,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f & \text{in } \Omega \\ u(0, t) = u_1 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial t}(0, t) = u_2 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\partial\Omega$ denotes the boundary of Ω, is a boundary problem.

The problem of finding a solution to an equation

$$\frac{dx}{dt} = f(t, x), \quad t \in J, \quad x \in \mathbf{R}^n,$$

lying in a given subset D of the space $D(J, \mathbf{R}^n)$ of functions depending on t that are absolutely continuous on J and that assume values in $\mathbf{R}^n$:

$$x(.) \in D.$$

It is assumed that $f(t, x)$ is a function defined on $J \times \mathbf{R}^n$ with values in $\mathbf{R}^n$ and satisfying the Carathéodory conditions; J is an interval on the real line $\mathbf{R}$.

1) The boundary value problem (1), (2) is said to be linear if

$$f(t, x) \equiv A(t)x + b(t),$$

where the functions $A(t)$ and $b(t)$ are summable on every compact interval in J and the set D is a linear manifold in $D(J, \mathbf{R}^n)$. In particular, one might have

$$J = [t_0, t_1],$$

$$D = \left\{ x(.) \in D(J, \mathbf{R}^n) : \int_{t_0}^{t_1} [d\,\Phi(t)]\, x(t) = 0 \right\},$$

where $\Phi(t)$ is a function of bounded variation. A linear boundary value problem gives rise to a linear operator

$$Lx(t) \equiv x' - A(t)x, \qquad x(.) \in D,$$

the eigen values of which are precisely those values of the parameter λ for which the homogeneous boundary value problem

$$x' - A(t)x = \lambda x, \qquad x(.) \in D,$$

has non-trivial solutions. These non-trivial solutions are the eigen functions of the operator L. If the inverse operator L^{-1} exists and has an integral representation

$$x(t) = L^{-1} b(t) \equiv \int_J G(t, s) b(s)\, ds, \qquad t \in J,$$

then $G(t, s)$ is called a Green function.

2) Let $J = (-\infty, \infty)$, let $f(t, x)$ be almost-periodic in t uniformly in x on every compact subset of $\mathbf{R}^n$ and let D be the set of almost-periodic functions in t that are absolutely continuous on J. Then problem (1), (2) is known as the problem of almost-periodic solutions.

3) In control theory one considers boundary value problems with a functional parameter: a control. For example, consider the equation

$$\frac{dx}{dt} = f(t, x, u), \qquad t \in J = [t_0, t_1], \qquad x \in \mathbf{R}^n,$$

with set of admissible controls U and two sets $M_0, M_1 \subset \mathbf{R}^n$. Let D be the set of absolutely continuous functions in t such that $x(t_0) \in M_0$, $x(t_1) \in M_1$. The boundary value problem is to find a pair $(x_0(.), u_0(.))$ such that $u_0(.) \in U$ and the solution $x_0(t)$ of equation (3) at $u = u_0(t)$ satisfies the condition $x_0(.) \in D$.

4) There is a wide range of diverse necessary and sufficient conditions for the existence and uniqueness of solutions to various boundary value problems, and of methods for constructing an approximate solution (see [4]–[7]). For example, consider the problem

$$\left. \begin{aligned} & x' = A(t)x + f(t, x), \\ & \int_{t_0}^{t_1} [d\,\Phi(t)]\, x(t) = 0, \end{aligned} \right\}$$

in which

$$\|f(t, x)\| \leq a + b\|x\|^{\alpha}$$

for certain constants $a > 0, b > 0, \alpha \geq 0$. Suppose that the homogeneous problem

$$x' = A(t)x, \qquad \int_{t_0}^{t_1} [d\,\Phi(t)]\, x(t) = 0$$

is regular, i.e. its only solution is the trivial one. Then problem (4) has at least one solution, provided either $\alpha < 1$, or $\alpha \geq 1$ and b is sufficiently small. It is fairly complicated to determine whether problem (5) is regular. However, the linear (scalar) boundary value problem

$$x'' + q(t)x' + p(t)x = 0, \qquad x(t_0) = 0, \qquad x(t_1) = 0,$$

for example, is regular if whenever $|q(t)| \leq 2m$ there exists a $k \in \mathbf{R}$ such that

$$\int_{t_0}^{t_1} [p(t) - k]_+ \, dt < 2\,[F(k, m) - m],$$

where

$$F(k,m)=\begin{cases}\sqrt{k-m^2}\,\mathrm{cotg}\,\dfrac{(t_1-t_0)\sqrt{k-m^2}}{2}, \\ \qquad m^2<k\le m^2+\dfrac{\pi^2}{(t_1-t_0)^2}, \\ \dfrac{2}{t_1-t_0}, \qquad k=m^2, \\ \sqrt{m^2-k}\,\mathrm{cotg}\,\dfrac{(t_1-t_0)\sqrt{m^2-k}}{2}, \qquad k<m^2.\end{cases}$$

11.4 BOUNDARY CONDITIONS

There are three types of boundary conditions commonly encountered in the solution of partial differential equations:

1. Dirichlet boundary conditions specify the value of the function on a surface $T=f(\mathbf{r},t)$.

2. Neumann boundary conditions specify the normal derivative of the function on a surface,

$$\frac{\partial T}{\partial n}=\hat{\mathbf{n}}\cdot\nabla T=f(\mathbf{r},t).$$

3. Robin boundary conditions. For an elliptic partial differential equation in a region Ω, Robin boundary conditions specify the sum of αu and the normal derivative of $u=f$ at all points of the boundary of Ω, with α and f being prescribed.

Check In Progress-I

Q. 1 State Boundary Value Problem.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

Q.2 Write Two Boundary Conditions.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

11.5 INITIAL VALUE PROBLEM

An initial value problem is a problem that has its conditions specified at some time $t = t_0$. Usually, the problem is an ordinary differential equation or a partial differential equation. For example,

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \nabla^2 u = f & \text{in } \Omega \\ u = u_0 & t = t_0 \\ u = u_1 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\partial\Omega$ denotes the boundary of Ω, is an initial value problem.

11.5.1 Solving BVPs

We are next interested in applying these theorems in the BVP

$$y'' = f(t, y, y')$$

$$y(0) = y(1) = 0$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. This includes, for example, the forced pendulum equation in the form

$$y'' + a \sin y = e(t)$$

$$y(0) = y(\frac{T}{2}) = 0$$

where $e(t)$ is t-periodic and odd. In this case, we take

$$f(t,u,v)=\frac{T^2}{4}(e(\frac{T}{2}t)-a\sin u),$$

where we have done the change of variable

$$t=\frac{T}{2}s;\ y(t)=u(s).$$

Write

$$Ly=y''.$$

(linear differential operator)

Then, $L:C^2[0,1]\to C[0,1]$. We introduce the *Nemitski Operator*, $F:C^1[0,1]\to C[0,1]$ given by

$$F(u)(t)=f(t,u(t),u'(t)).$$

Let j be the inclusion operator from C^2 to C^1. To solve the boundary value problem, we need to invert L and apply it to $Ly=f(t,y,y')$. First, we need to restrict L to the subspace

$$C_0^2[0,1]=C_0^2=\{u\in C^2:\ u(0)=u(1)=0\};$$

then we will have the solution to the BVP written as a solution to the operator equation,

$$y=(L^{-1}\circ F\circ j)(y)$$

Define $S:=L^{-1}\circ F\circ j:C_0^2\to C_0^2$.

Recall the Green's function for the problem

$$u''=0,\ u(0)=u(1)=0,$$

given by

$$G(s,t):=\begin{cases}(t-1)s, & 0\le s\le t\le 1\\ t(s-1), & 0\le t\le s\le 1\end{cases}$$

Recall,

Lemma 6.1 *For* $w \in C$, $u = L^{-1}w$ *is given by*

$$u(t) = \int_0^1 G(s,t)w(s)ds$$

and satisfies

$$u''(t) = w,\ u(0) = u(1) = 0.$$

Proof. Verify this directly (we've seen it before). Λ

We also have,

Lemma 6.2 *For* $w \in C$,

$$\|L^{-1}w\|_2 \leq \frac{7}{4}\|w\|$$

Proof. Recall for $u \in C^2$, the norm

$$\|u\|_2 = \max|u(t)| + \max|u'(t)| + \max|u''(t)|$$

$$= \|u\| + \|u'\| + \|u''\|$$

Notice that

$$G(s,t) \leq 0$$

and it is easy to see that

$$-\frac{1}{4} \leq G(s,t) \leq 0.$$

So letting $u = L^{-1}w$,

$$|u(t)| = |L^{-1}w(t)| \leq \int_0^1 |G(t,s)||w(s)|ds \leq \frac{1}{4}\|w\|.$$

Now,

$$u'(t) = \int_0^t sw(s)ds + \int_t^1 (s-1)w(s)ds$$

$$|u'(t)| = \left|\int_0^t sw(s)ds + \int_t^1 (s-1)w(s)ds\right|$$

$$\leq \|w\| \left(\int_0^t s ds + \int_t^1 (s-1) ds \right)$$

$$\leq \|w\| \left(\frac{t^2}{2} + \frac{(1-t)^2}{2} \right) \leq \frac{1}{2} \|w\|$$

Finally, through similar estimates, we get $\|u''(t)\| = \|w(t)\| \leq \|w\|$, so

$$\|L^{-1}w\|_2 = \|u\|_2 \leq \left(\frac{1}{4} + \frac{1}{2} + 1 \right) \|w\|$$

$$= \frac{7}{4} \|w\|$$

Hence, as L^{-1} has this uniform bound, we get that it is continuous (from a simple proof in functional analysis). Λ

Lemma 6.3 *If* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous, then the Nemitski operator* $F : C^1[0,1] \to C[0,1]$ *is continuous in* C^1*, where*

$$(Fu)(t) := f(t, u(t), u'(t))$$

Proof. Let $u \in C^1[0,1]$, and let $\epsilon > 0$ be given. We need to show there exists $\delta = \delta(\epsilon, u) > 0$ such that if $v \in C^1[0,1]$ and $\|u - v\|_1 < \delta$, then $\|F(v) - F(u)\| < \epsilon$. Chose $r \geq 1$ such that $\|u\|_1 \leq r$. Now, on the compact set

$$Q := [0,1] \times [-2r, 2r] \times [-2r, 2r],$$

f is bounded and uniformly continuous. So, there exists a $\hat{\delta} > 0$ such that if $|t_1 - t_2| + |u_1 - u_2| + |v_1 - v_2| < \hat{\delta}$, then

$$|f(t, u_1, v_1) - f(t, u_1, v_1)| < \epsilon, \forall (t_i, u_i, v_i) \in Q, i = 1,2$$

Let $\delta = \min\{\hat{\delta}, r\}$. If $v \in C^1[0,1]$ and $\|u0v\|_1 < \delta$, then

$$\|v\|_1 \leq \|v - u\|_1 + \|u\|_1 < \delta + r \leq 2r.$$

Hence,

$|v(t)| < 2r,$ and $|v'(t)| < 2r$

So, $(t, u(t), u'(t)) \in Q$ and $(t, v(t), v'(t)) \in Q$.

Hence, $|f(t, u(t), u'(t)) - f(t, v(t), v'(t))| < \epsilon$ for all t and $\|F(u) - F(v)\| < \epsilon$. Thus, $F : C^1[0,1] \to C[0,1]$ is continuous.

Λ

We now have:

Theorem 6.1 *The operator*

$S : C_0^2[0,1] \to C_0^2[0,1],$

given by,

$(S(y))(t) = L^{-1}(F(j(y))(t)$

is completely continuous.

Proof. Immediate. $j : C^2[0,1] \to C^1[0,1]$ given by $j(u) = u$ is completely continuous. Then S is also, as it is the composition of a continuous and completely continuous map. Λ

We now easily prove:

Theorem 6.2 *If* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is bounded, then the BVP*

$y'' = f(t, y, y')$

$y(0) = y(1) = 0$

has a solution.

Proof. Let

$m := \sup\{|f(t, u, v)| : (t, u, v) \in [0,1] \times \mathbb{R}^2\};$

$B := \{u \in C_0^2[0,1] : \|u\|_2 \leq \frac{7}{4}m\}.$

where

$\|u\|_2 = \max |u(t)| + \max |u'(t)| + \max |u''(t)|.$

Notes

Then we note that B is closed, bounded, and convex. We claim $S(B) \subset B$. Let $u \in B$; then we want to show that $S(u) \in B$. We note that

$$\|F(j(u))\| \leq m.$$

i.e.,

$$|f(t, u(t), u'(t)) \leq m, \forall t \in [0,1].$$

Let $S = L^{-1} \circ F \circ j$. Then,

$$\|Su\|_2 = \|L^{-1}(F(j(u)))\|_2 \leq \frac{7}{4}\|F(j(u))\| \leq \frac{7}{4}m$$

Thus,

$$S(u) \in B$$

and S is a completely continuous operator in B into B, and so has a fixed point $u \in B$ with $S(u) = u = L^{-1}(F(j(u)))$. Then,

$$Lu = F(j(u)) = f(u, u, u')$$

$$u(0) = u(1) = 0$$

and the BVP has a solution. Λ

We get this as an immediate Corollary:

Theorem 6.3 *Suppose* $e : \mathbb{R} \to \mathbb{R}$ *is an odd* T*-periodic function. Then the BVP*

$$y'' = a \sin y = e,$$

$$y(0) = y(\frac{T}{2}) = 0,$$

has a solution. By extending y *to an odd function and extending, we can find a periodic solution of the BVP for all of* $\mathbb{R}$.

11.6 TOPICS IN NONLINEAR BVPS

Consider

$$x'' = f(t, x, x') \tag{11}$$

$$x(t_1) = x_1,\ x(t_2) = x_2 \tag{12}$$

$f : J \times \mathbb{R}^n \to \mathbb{R}^n,\ J = [a, b] \subset \mathbb{R},\ t_1, t_2 \in J$

$x \in \mathbb{R}^n, \|x\|$ - any norm

Recall that the only solution to $x'' = 0$ satisfying $x(t_1) = x(t_2) = 0$ is $x \equiv 0$, so for each $h \in C(J, \mathbb{R}^n)$, the unique solution of

$$x'' = h(t),\ x(t_1) = x(t_2) = 0$$

is given by

$$x(t) = \int_{t_1}^{t_2} G(t, s) h(s) ds$$

where $G(t, s)$ is given by

$$G(t, s) = \begin{cases} \frac{1}{t_1 - t_2}(t_2 - t)(s - t_1), & s \le t \\ \frac{1}{t_1 - t_2}(t_2 - s)(t - t_1), & t \le s \end{cases}$$

Therefore the solution to

$$x'' = h(t),\ x(t_1) = x_1,\ x(t_2 = x_2$$

is given by

$$x(t) = \int_{t_1}^{t_2} G(t, s) h(s) ds + w(t)$$

where

$$w'' = 0,\ w(t_1) = x_1,\ w(t_2) = x_2$$

And the solution of the original BVP (11), (12) is given by

$$x(t) = \int_{t_1}^{t_2} G(t, s) f(s, x(s), x'(s)) ds + w(t) \tag{13}$$

Consequently, $x(t)$ solves (13) if and only if $x(t)$ solves (11), (12).

Recall some properties of the Green's function. For fixed t,

$$\max_{t_1 \le s \le t_2} |G(t,s)| = |G(t,t)|$$

$$\max_{t_1 \le t \le t_2} |G(t,t)| = \frac{t_2 - t_1}{4}$$

Therefore,

$$|G(t,s)| \le \frac{t_2 - t_1}{4},\ s,t \in [t_1, t_2]$$

and

$$\int_{t_1}^{t_2} |G(t,s)| ds = \frac{(t_2 - t)(t - t_1)}{2}$$

and so,

$$\int_{t_1}^{t_2} |G(t,s)| ds \le \frac{(t_2 - t_1)^2}{8}$$

Also, we have

$$\int_{t_1}^{t_2} |G_t(t,s)| ds \le \frac{(t - t_1)^2 + (t_2 - t)^2}{2(t_2 - t_1)}$$

and the max occurs at either $t = t_1$ or $t = t_2$, so

$$\int_{t_1}^{t_2} |G_t(t,s)| ds \le \frac{(t_2 - t_1)}{2}.$$

Theorem 7.1 *Assume* $\exists K, L > 0$ *such that for all* $(t, x_1, y_1), (t, x_2, y_2) \in J \times \mathbb{R}^n \times \mathbb{R}^n$*, we have*

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \le K\|x_1 - x_2\| + L\|y_1 - y_2\|$$

Then, if K, L *satisfies*

$$K\frac{(t_2-t_1)^2}{8}+L\frac{t_2-t_1}{2}<1,$$

the BVP (11), (12) has a unique solution.

Proof.

$$B:=\{u\in C^1(J,\mathbb{R}^n)$$

and for $u\in B$, we define

$$\|u\|_B=\max_{t_1\le t\le t_2}(K\|u(t)\|+L\|u'(t)\|)$$

Then, define the operator $T:B\to B$ by

$$(Tu)(t)=\int_{t_1}^{t_2}G(t,s)f(s,u(s),u'(s))ds+w(t)$$

then we can verify for $u_1,u_2\in B$,

$$\|(Tu_2)(t)-(Tu_1)(t)\|\le\frac{(t_2-t_1)^2}{8}[K\|u_2(t)-u_1(t)+L\|u_2'(t)-u_1'(t)\|]$$

$$\le\frac{(t_2-t_1)^2}{8}\|u_2-u_1\|_B$$

Since

$$(Tu)'(t)=\int_{t_1}^{t_2}G_t(t,s)f(s,u(s),u'(s))ds,$$

we get:

$$(Tu_2)'(t)-(Tu_1)'(t)\|\le\|\int_{t_1}^{t_2}G_t(t,s)[f(s,u_2(s),u_2'(s))-f(s,u_1(s),u_1'(s))]$$

$$\le\int_{t_1}^{t_2}|G_t(t,s)|[K\|u_2(s)-u_1(s)\|+L\|u_2'(s)-u_1'(s)\|]ds$$

$$\le\max_s[K\|u_2(s)-u_1(s)\|+L\|u_2'(s)-u_1'(s)\|]\int_{t_1}^{t_2}|G_t(t,s)|ds\le\frac{t_2-t_1}{2}\|u_2-u_1\|_B$$

Therefore,

$$\|Tu_2-Tu_1\|_B\le\left(K\frac{(t_2-t_1)^2}{8}+L\frac{t_2-t-1}{2}\right)\|u_2-u_1\|_B$$

Notes

Hence, we can apply the Contraction Mapping Principle to conclude that there exists a fixed point of T; hence, the BVP (11), (12) has a solution. Λ

Remark: In the scalar case, we can try to show that this result is sharp as follows:

Suppose that $f \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ and satisfies

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq K\|x_1 - x_2\| + L\|y_1 - y_2\|$$

let $u(t)$ be the solution of

$$u'' + Lu' + Ku = 0$$

$$u(t_1) = 0, u'(t_1) = 1$$

Then, we can show that there exists $\hat{t} = \hat{t}(L, K) > t_1$ such that $u'(\hat{t}) = 0$ and $u'(t) > 0$ on $[t_1, \hat{t})$.

But we can also show that the BVP (11), (12) has a unique solution if

$$\frac{t_2 - t_1}{2} < \hat{t} - t_1$$

This is best possible.

Next, we will be interested in solving the BVP (11), (12) for $t_2 - t_1$ small, but without uniqueness.

Theorem 7.2 *Le* $M, N > 0$ *be given and assume* $f \in C[J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ *and let*

$$q := \max\{\|f(t, x, x')\| : t \in J, \|x\| \leq 2M, \|x'\| \leq 2N\}$$

Define

$$\delta := \min\{\sqrt{\frac{8M}{q}}, \frac{2N}{q}\}$$

then the BVP (11),(12) has a solution for any $t_1, t_2 \in J$, $\|x_1\| \leq M$, $\|x_2\| \leq M$,

$$\left\|\frac{x_1 - x_2}{t_1 - t_2}\right\| \leq N,$$

and $0 < t_2 - t_1 \leq \delta$*. Moreover, given* $\epsilon > 0$*, there exists a solution* $x(t)$ *such that*

$$\|x(t) - w(t)\| < \epsilon$$

and

$$\|x'(t) - w'(t)\| < \epsilon$$

on $[t_1, t_2]$*, provided* $t_2 - t_1$ *is small enough.*

Proof. We apply the Schauder fixed-point theorem. Let

$$B = C'([t_1, t_2], \mathbb{R}^n),$$

and define the norm

$$\|x\|_B := \max_{[t_1,t_2]} \|x(t)\| + \max_{[t_1,t_2]} \|x'(t)\|$$

Let

$$B_0 := \{x \in B : \|x\| \leq 2M, \|x'\| \leq 2N\}$$

(here, $\|x(t)\| := \max \|x(t)\|$). B_0 is closed, convex, and we define $T : B \to B$ by

$$(Tx)(t) := \int_{t_1}^{t_2} G(t,s) f(s, x(s), x'(s)) ds + w(t)$$

For $x \in B_0$,

$$\|(Tx)(t)\| \leq \| \int_{t_1}^{t_2} G(t,s) f(s, x(s), x'(s)) ds\| + \|w(t)\|$$

$$\leq \left(\max_t \int_{t_1}^{t_2} |G(t,s)| ds\right) \max_{B_0} \|f(t, x, x')\| + \max_t \|w(t)\|$$

$$\leq q\left(\frac{(t_2 - t_1)^2}{8}\right) + M$$

Also,

$$\|(Tx)'(t)\| \leq q\left(\frac{t_2 - t_1}{2}\right) + N,$$

so if $0 \le t_2 - t_1 \le \delta$, then $T : B_0 \to B_0$. Also,

$$\|(Tx)''\| = \|f(t, x, x')\| \le q$$

So, $T : B_0 \to B_0$ is completely continuous, and hence has a fixed point by Schauder. Finally,

$$\|x(t) - w(t)\| \le \frac{q(t_2 - t_1)^2}{8} < \epsilon$$

$$\|x'(t) - w'(t)\| \le \frac{q(t_2 - t_1)}{2} < \epsilon$$

Λ

Theorem 7.3 *Assume* $f \in C([t_1, t_2] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ *and is bounded on* $[t_1, t_2] \times \mathbb{R}^n \times \mathbb{R}^n$*. Then, every BVP (11), (12) has a solution.*

Proof. Let

$$q := \sup\{\|f(t, x, y)\| : t_1 \le t \le t_2, x, y \in \mathbb{R}^n\}$$

and $q < \infty$. Given x_1, x_2, choose $M > 0$ such that $\|x_1\| \le M$, $\|x_2\| \le M$ and

$$\frac{\|x_2 - x_1\|}{t_2 - t_1} \le M$$

and

$$t_2 - t_1 \le \frac{8\sqrt{M}}{q}$$

$$t_2 - t_1 \le \frac{2M}{q}$$

Check In Progress-II

Q. 1 *The operator*

$$S : C_0^2[0, 1] \to C_0^2[0, 1],$$

given by,

$$(S(y))(t) = L^{-1}(F(j(y))(t)$$

is completely continuous.

Solution : . . . .

Q.2 *Assume* $f \in C([t_1, t_2] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ *and is bounded on* $[t_1, t_2] \times \mathbb{R}^n \times \mathbb{R}^n$*. Then, every BVP (11), (12) has a solution.*

Solution : . . . .

11.6.1 Differential Inequalities in Boundary Value Problems

Definition 7.2.1 *A function* $\alpha \in C(J, \mathbb{R}) \cap C^2(J^\circ, \mathbb{R})$ *is a lower solution of the DE*

$$x'' = f(t, x, x')$$

in case that $\alpha'' \geq f(t, \alpha, \alpha')$ *for* $t \in J^\circ$*. Further,* β *is an upper solution if*

$$\beta'' \leq f(t, \beta, \beta'), t \in J^\circ$$

Here, we are assuming that $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Lemma 7.2.1 *Assume* $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and is non-decreasing in* x *for each fixed* (t, y) *in* $J \times \mathbb{R}$*. If* α, β *are lower and upper solutions, respectively, with*

$\alpha(a) \le \beta(a), \alpha(b) \le \beta(b)$

and if one of the differential inequalities is strict, then

$\alpha(t) \le \beta(t),\ t \in J^\circ$

Proof. Suppose that $\alpha(t) \ge \beta(t)$ for some $t \in J^\circ$. Then with

$m(t) = \alpha(t) - \beta(t),$

$m(t)$ has a nonnegative max at some $t_0 \in (a,b) = J^\circ$. Of course, $m(t_0) \ge 0$, $m'(t_0) = 0$, and $m''(t_0) \le 0$. Then,

$\alpha(t_0) \ge \beta(t_0)$

$\alpha'(t_0) = \beta'(t_0)$

$\alpha''(t_0) \ge \beta''(t_0)$

Therefore,

$\alpha''(t_0) \ge f(t_0, \alpha(t_0), \alpha'(t_0)) \ge f(t_0, \beta(t_0), \beta'(t_0)) \ge \beta''(t_0)$

Since one of the inequalities is strict, we get a contradiction. Recall that we are considering

$$x'' = f(t, x, x') \tag{14}$$

We are going to modify the RHS f so it is bounded in all of $[a,b] \times \mathbb{R}^2$.

Suppose that $\alpha, \beta \in C[a,b]$ and that $\alpha(t) \le beta(t)$ on $J = [a,b]$.

Let $C > 0$ be such that

$|\alpha'(t)| < C,\ |\beta'(t)| < C$ on J

We define

$$F^*(t,x,x') := \begin{cases} f(t,x,C), & x' \ge C \\ f(t,x,x'), & |x'| \le C \\ f(t,x,-C), & x' \le -C \end{cases}$$

and we define

$$F(t,x,x') := \begin{cases} F^*(t,\beta(t),x') + \frac{x-\beta(t)}{1+x^2}, & x > \beta(t) \\ F^*(t,x,x'), & \alpha(t) \le x \le \beta(t) \\ F^*(t,\alpha(t),x') + \frac{x-\alpha(t)}{1+x^2}, & x < \alpha(t) \end{cases}$$

We call $F(t,x,x')$ the modification of $f(t,x,x')$ with respect to (α,β,C). From the definition of F, we have

$|F(t,x,x')| \le M$ on $J \times \mathbb{R}^2$

where $M = M_0 + 1$ and

$M_0 = \max\{|f(t,x,x')| : t \in J, \alpha(t) \le x \le \beta(t), |x'| \le C\} + \max|\alpha(t)| + \max_J |\beta(t)|$

We have the fact that

$$\left|\frac{x-\beta(t)}{1+x^2}\right| \le \frac{|x|}{1+x^2} + \frac{|\beta(t)|}{1+x^2} \le 1 + |\beta(t)|$$

We consider the BVP

$$x'' = F(t,x,x') \qquad (15)$$

$$x(a) = \gamma,\ x(b) = \delta \qquad (16)$$

We have then

Theorem 7.2.2 *Let* $\alpha,\beta \in C(J,\mathbb{R}) \cap C^2(J^o,\mathbb{R})$ *be lower and upper solutions of (14) with* $\alpha(t) \le beta(t)$ *on* J. *Then the modified BVP (15), (16) has a solution for any* $\alpha(a) \le \gamma\beta(a)$, $\alpha(a) \le \delta \le \beta(b)$ *and such that* $\alpha(t) \le x(t) \le \beta(t)$.

Proof. There is a solution to (15), (16) because F is bounded. We need only toshow that the solution stays between $\alpha(t)$ and $\beta(t)$. If the solution $x(t) > \beta(t)$ for some t, then $m(t) = x(t) - \beta(t)$ has a max at $t_0 \in J$, $m'(t_0) = 0$, and $m''(t_0) \le 0$. This eventually leads to a contradiction.

Λ

Definition 7.2.2 *Let* $f \in C[J \times \mathbb{R}^2, \mathbb{R})$ *and assume* $\alpha, \beta \in C(J, \mathbb{R})$ *with* $\alpha(t) \leq beta(t)$ *on* J. *We say that* f *satisfies a Nagumo condition with respect to* α, β *on* J *if there exists an* η *such that*

$$h(|x'|) > 0, \forall 0 \leq |x'| \leq +\infty$$

and such that

$$|f(t, x, x')| \leq h(|x'|)$$

for all $t \in J$, $\alpha(t) \leq x(t) \leq \beta(t)$, *and for all* x'. h *must also satisfy*

$$\int_{\lambda}^{\infty} \frac{s\,ds}{h(s)} > \max \beta(t) - \min \alpha(t) \qquad (17)$$

and where

$$\lambda = \frac{1}{b-a} \max\{|\alpha(a) - \beta(b)|, |\alpha(b) - \beta(a)|\}$$

Note that if $\int_{c}^{\infty} \frac{s}{h(s)} = +\infty$, then (17) holds.

Theorem 7.2.3 *Suppose* $f \in C[J \times \mathbb{R}^2, \mathbb{R})$ *satisfies a Nagumo condition with respect to the pair* $\alpha, \beta \in C(J, \mathbb{R})$. *Then for any solution* $x \in C^2(J, \mathbb{R})$ *of* $x'' = f(t, x, x')$ *with* $\alpha(t) \leq x(t) \leq \beta(t)$ *on* J, *there exists an* $N > 0$ *depending only on* α, β, h *such that* $|x'(t)| \leq N$ *on* J.

Proof. We leave this unproven, but it is shown in Kelley and Peterson.

Choose $N > \lambda$ such that

$$\int_{\lambda}^{N} \frac{s\,ds}{h(s)} > \max \beta(t) - \min \alpha(t)$$

Let $t_0 \in J^{\circ}$ such that

$$x'(t_0) = \frac{x(b) - x(a)}{b - a}$$

and $|x'(t_0)| \le \lambda$. Claim that $|x'(t)| \le N$ for all $t \in J$. If not, then there exists an interval $[t_1, t_2] \subset J$ such that one of the following holds:

1. $x'(t_1) = \lambda, x'(t_2) = N, \lambda < x' < N$ on (t_1, t_2)
2. $x'(t_1) = N, x'(t_2) = \lambda, N < x' < \lambda$ on (t_1, t_2)
3. $x'(t_1) = -\lambda, x'(t_2) = -N, -N < x' < -\lambda$ on (t_1, t_2)
4. $x'(t_1) = -N, x'(t_2) = -\lambda, -N < x' < -\lambda$ on (t_1, t_2)

Consider 1. Then, we have

$$|x''(t)|x'() = |f(t, x, x')|x'(t) \le h(|x'|)x'(t)$$

So,

$$\left| \int_{t_1}^{t_2} \frac{x''(s)x'(s)}{h(|x'(s)|)} ds \right| \le \int_{t_1}^{t_2} \frac{|x''(s)|x'(s)}{h(x'(s))} \le \int_{t_1}^{t_2} x'(s) ds$$

$$x(t_2) - x(t_1) \le \max \beta(t) - \min \alpha(t)$$

On the other hand,

$$\tau = |x'(s)|$$

$$d\tau = x''(s) ds$$

$$\left| \int_{t_1}^{t_2} \frac{x''(s)x'(s)}{h(|x'(s)|)} ds \right| = \int_{\lambda}^{N} \frac{\tau d\tau}{h(\tau)} > \max \beta(t) - \min \alpha(t)$$

Hence,

$$|x'(t)| \le N$$

on J. Λ

Theorem 7.2.4 *If* $\int^{\infty} \frac{s\,ds}{h(s)} = \infty$, *then the conclusion of Theorem* **7.2.3** *holds.*

Remark: This does not hold for systems. To see this, let $x_n(t) = (\cos nt, \sin nt)$, $\|x_n\| = 1$ Then,

$$\|x_n'\| = n$$

$$\|x_n''\| = n^2$$

then, the hypotheses of the last theorem hold with $h(s) = s^2 + 1$.

11.7 LET'S SUM UP

We study in this unit A boundary value problem is a problem, typically an ordinary differential equation or a partial differential equation, which has values assigned on the physical boundary of the domain in which the problem is specified. For example,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f & \text{in } \Omega \\ u(0, t) = u_1 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial t}(0, t) = u_2 & \text{on } \partial\Omega, \end{cases}$$

where $\partial\Omega$ denotes the boundary of Ω, is a boundary problem.

We also study following:

1. Fixed Point Theorem of cone expansion and cone compression. Definition. How these are applied ($\frac{f(u)}{u} \to 0$ at end points).
2. Review some homework examples.
3. Application of L and L^{-1} with respect to solving the periodic BVP. (Forced pendulum equation). Formulation of BVP.
4. Fredholm operator of index zero.

5. Nagumo condition.

11.8 KEYWORD

Boundary Value : a value specified by a boundary condition

Fredholm Operator : a Fredholm operator is an operator that arises in the Fredholm theory of integral equations. ... A Fredholm operator is a bounded linear operator T : X → Y between two Banach spaces with finite-dimensional kernel and (algebraic) cokernel , and with closed range.

Sturm–Liouville problems : In mathematics and its applications, a classical *Sturm–Liouville* theory, named after Jacques .... The eigenvalues λ1, λ2, λ3, ... of the regular *Sturm–Liouville problem* (1)–(2)–(3) are real and can be ordered such that ..... In this space L is *defined* on sufficiently smooth functions which satisfy the above boundary conditions

11.9 QUESTIONS FOR REVIEW

Q. 1 *Let* $\alpha, \beta \in C(J,\mathbb{R}) \cap C^2(J^\circ,\mathbb{R})$ *be lower and upper solutions of (14) with* $\alpha(t) \leq beta(t)$ *on* J*. Then the modified BVP (15), (16) has a solution for any* $\alpha(a) \leq \gamma\beta(a)$, $\alpha(a) \leq \delta \leq \beta(b)$ *and such that* $\alpha(t) \leq x(t) \leq \beta(t)$.

Q. 2 *For* $w \in C$, $u = L^{-1}w$ *is given by*

$$u(t) = \int_0^1 G(s,t)w(s)ds$$

and satisfies

$$u''(t) = w,\ u(0) = u(1) = 0.$$

Q. 3 *If* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous, then the Nemitski operator* $F : C^1[0,1] \to C[0,1]$ *is continuous in* C^1*, where*

Notes

$(Fu)(t) := f(t, u(t), u'(t))$

Q. 4 *Assume* $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and is non-decreasing in* x *for each fixed* (t, y) *in* $J \times \mathbb{R}$. *If* α, β *are lower and upper solutions, respectively, with*

$\alpha(a) \le \beta(a), \alpha(b) \le \beta(b)$

and if one of the differential inequalities is strict, then

$\alpha(t) \le \beta(t), t \in J^{\circ}$

Q. 5 Define Boundary Value Problem for Second Order Differential Equation.

11.10 SUGGESTION READING AND REFERENCES

[1] P. Hartman, "Ordinary differential equations" , Birkhäuser (1982)

[2] M.A. Krasnosel'skii, V.Sh. Burd, Yu.S. Kolesov, "Nonlinear almost-periodic oscillations" , Wiley (1973) (Translated from Russian)

[3] L.S. Pontryagin, V.G. Boltayanskii, R.V. Gamkrelidze, E.F. Mishchenko, "The mathematical theory of optimal processes" , Interscience (1962) (Translated from Russian)

[4] N.N. Krasovskii, "Theory of control of motion. Linear systems" , Moscow (1968) (In Russian)

[5] V.I. Zubov, "Lectures in control theory" , Moscow (1975) (In Russian)

[6] E. Kamke, "Differentialgleichungen: Lösungen und Lösungsmethoden" , 1. Gewöhnliche Differentialgleichungen , Chelsea, reprint (1971)

[7] I.T. Kiguradze, "Some singular boundary value problems for ordinary

differential equations" , Tbilisi (1975) (In Russian)

11.11 ANSWER TO CHECK IN PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 4

Q. 2 Check in Section 5

Check In progress-II

Answer Q. 1 Check in Section 6.1

Q. 2 Check in Section 6.1

UNIT 12: GREEN’S FUNCTION

STRUCTURE

12.0 OBJECTIVE

- In this unit we study Green’s Function and its examples
- We also study A Green’s Function of a linear Differential operator.
- WE LEARN GREEN'S FUNCTION--POISSON'S EQUATION AND ITS EXAMPLES.
- We also learn Helmholtz Differential Equation and Green Function for Ordinary Differential Equations
- We study Green function for partial differential equations and Fredohlm Operator

12.1 INTRODUCTION

A function related to integral representations of solutions of boundary value problems for differential equations.

The Green function of a boundary value problem for a linear differential equation is the fundamental solution of this equation satisfying homogeneous boundary conditions. The Green function is the kernel of the integral operator inverse to the differential operator generated by the given differential equation and the homogeneous boundary conditions (cf. Kernel of an integral operator). The Green function yields solutions of the inhomogeneous equation satisfying the homogeneous boundary conditions. Finding the Green function reduces the study of the properties of the differential operator to the study of similar properties of the corresponding integral operator.

In mathematics, a **Green's function** of an inhomogeneous linear differential operator defined on a domain with specified initial conditions or boundary conditions is its impulse response.

This means that if L is the linear differential operator, then

- the Green's function G is the solution of the equation $LG = \delta$, where δ is Dirac's delta function;
- the solution of the initial-value problem $Ly = f$ is the convolution ($G * f$), where G is the Green's function.

Through the superposition principle, given a linear ordinary differential equation (ODE), L(solution) = source, one can first solve L(green) $= \delta_s$, for each s, and realizing that, since the source is a sum of delta functions, the solution is a sum of Green's functions as well, by linearity of L.

Green's functions are named after the British mathematician George Green, who first developed the concept in the 1830s. In the modern study of linear partial differential equations, Green's functions are studied largely from the point of view of fundamental solutions instead.

Under many-body theory, the term is also used in physics, specifically in quantum field theory, aerodynamics, aeroacoustics, electrodynamics, seismology and st atistical field theory, to refer to various types of correlation functions,

Notes

even those that do not fit the mathematical definition. In quantum field theory, Green's functions take the roles of propagators.

12.2 DEFINITION AND USES

A Green's function, *G*(*x*,*s*), of a linear differential operator L = L(x) acting on distributions over a subset of the Euclidean space Rn, at a point *s*, is any solution of

$$L \quad G(x, s) = \delta(s - x)$$

where δ is the Dirac delta function. This property of a Green's function can be exploited to solve differential equations of the form

$$L\, u(x) = f(x)$$

If the kernel of *L* is non-trivial, then the Green's function is not unique. However, in practice, some combination of symmetry, boundary conditions and/or other externally imposed criteria will give a unique Green's function. Green's functions may be categorized, by the type of boundary conditions satisfied, by a Green's function number. Also, Green's functions in general are distributions, not necessarily functions of a real variable.

Green's functions are also useful tools in solving wave equations and diffusion equations. In quantum mechanics, the Green's function of the Hamiltonian is a key concept with important links to the concept of density of states.

The Green's function as used in physics is usually defined with the opposite sign, instead. That is,

$$L \quad G(x, s) = \delta(s - x)$$

This definition does not significantly change any of the properties of the Green's function due to the evenness of the Dirac delta function.

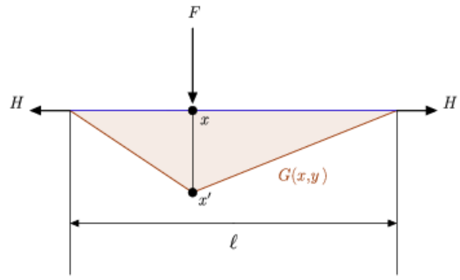
$$G(x, s) = G(s - x)$$

If the operator is translation invariant, that is, when L has constant coefficients with respect to *x*, then the Green's function can be taken to

be a convolution kernel, that is,

In this case, the Green's function is the same as the impulse response of linear time-invariant system theory.

12.3 GREEN'S FUNCTION



Generally speaking, a Green's function is an integral kernel that can be used to solve differential equations from a large number of families including simpler examples such as ordinary differential equations with initial or boundary value conditions, as well as more difficult examples such as inhomogeneous partial differential equations (PDE) with boundary conditions. Important for a number of reasons, Green's functions allow for visual interpretations of the actions associated to a source of force or to a charge concentrated at a point, thus making them particularly useful in areas of applied mathematics. In particular, Green's function methods are widely used in, e.g., physics, and engineering.

More precisely, given a linear differential operator $\mathcal{L} = \mathcal{L}(x)$ acting on the collection of distributions over a subset Ω of some Euclidean space $\mathbb{R}^n$, a Green's function $G = G(x, s)$ at the point $s \in \Omega$ corresponding to $\mathcal{L}$ is any solution of

$$\mathcal{L} G(x, s) = \delta(x - s) \qquad (1)$$

where δ denotes the delta function. The motivation for defining such a function is widespread, but by multiplying the above identity by a function $f(s)$ and integrating with respect to s yields

Notes

$$\int \mathcal{L} G(x,s) f(s)\, ds = \int \delta(x-s) f(s)\, ds. \tag{2}$$

The right-hand side reduces merely to $f(x)$ due to properties of the delta function, and because $\mathcal{L}$ is a linear operator acting only on x and not on s, the left-hand side can be rewritten as

$$\mathcal{L}\left(\int G(x,s) f(s)\, ds\right). \tag{3}$$

This reduction is particularly useful when solving for $u = u(x)$ in differential equations of the form

$$\mathcal{L} u(x) = f(x), \tag{4}$$

where the above arithmetic confirms that

$$\mathcal{L} u(x) = \mathcal{L}\left(\int G(x,s) f(s)\, ds\right) \tag{5}$$

and whereby it follows that u has the specific integral form

$$u(x) = \int G(x,s) f(s)\, ds. \tag{6}$$

The figure above illustrates both the intuitive physical interpretation of a Green's function as well as a relatively simple associated differential equation with which to compare the above definition. In particular, it shows a taut rope of length ℓ suspended between two walls, held into place by an identical horizontal force H applied on each of its ends, and a lateral load F placed at some interior point x on the rope. Let x' be the point corresponding to x on the deflected rope, suppose the downward force F is constant, say $F = 1$, and let $u(x)$ denote the deflection of the rope. Corresponding to this physical system is the differential equation

$$-H u''(x) = F(x) \tag{7}$$

for $0 < x < \ell$ with $u(0) = u(\ell) = 0$, a system whose simplicity allows both its solution $u(x)$ and its Green's function $G(x, y)$ to be written explicitly:

$$u(x) = \frac{F}{2H}\left(\ell x - x^2\right) \tag{8}$$

and

$$G(x, y) = \frac{1}{H\ell}\begin{cases} y(\ell - x) & \text{for } y \le x \\ x(\ell - y) & \text{for } x \le y, \end{cases} \quad (9)$$

respectively. As demonstrated in the above figure, the displaced rope has the piecewise linear format given by $G = G(x, y)$ above, thus confirming the claim that the Green's function G associated to this system represents the action of the horizontal rope corresponding to the application of a force F.

A Green's function taking a pair of arguments (x, s) is sometimes referred to as a two-point Green's function. This is in contrast to multi-point Green's functions which are of particular importance in the area of many-body theory.

As an elementary example of a two-point function as defined above, consider the problem of determining the potential $\psi(\mathbf{r})$ generated by a charge distribution whose charge density is $\rho(\mathbf{r})$, whereby applications of Poisson's equation and Coulomb's law to the potential at $\mathbf{r}_1$ produced by each element of charge $\rho(\mathbf{r}_2)\, d^3\,\mathbf{r}_2$ yields a solution

$$\psi(\mathbf{r}_1) = \frac{1}{4\pi\varepsilon_0}\int d^3\,\mathbf{r}_2\,\frac{\rho(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \quad (10)$$

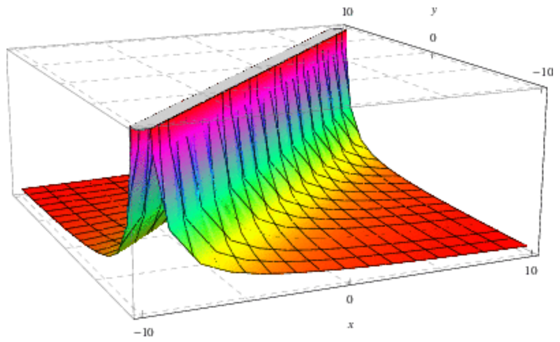
which holds, under certain conditions, over the region where $\rho(\mathbf{r}_2) \neq 0$. Because the right-hand side can be viewed as an integral operator converting ρ into ψ, one can rewrite this solution in terms of a Green's function $G = G(\mathbf{r}_1, \mathbf{r}_2)$ having the form

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}, \quad (11)$$

whereby the solution can be rewritten:

$$\psi(\mathbf{r}_1) = \int d^3\,\mathbf{r}_2\, G(\mathbf{r}_1, \mathbf{r}_2)\,\rho(\mathbf{r}_2) \quad (12)$$

Notes



The above figure shows the Green's function associated to the solution of the ψ - ρ equation discussed above where here, $\varepsilon_0 = 4$ and $\mathbf{r}_1$, respectively $\mathbf{r}_2$, is plotted on the x-, respectively y-, axis.

A somewhat comprehensive list of Green's functions corresponding to various differential equations is maintained online by Kevin Cole (Cole 2000).

Due to the multitude of literature written on Green's functions, several different notations and definitions may emerge, some of which are topically different than the above but which in general do not affect the important properties of the results. As the above example illustrates, for instance, some authors prefer to denote the variables x and s in terms of vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ to emphasize the fact that they're elements of $\mathbb{R}^n$ for some n which may be larger than 1 (Arfken 1985). It is also relatively common to see the definition with a negative sign so that G is defined to be the function for which

$$\mathcal{L}\, G(x, s) = -\delta(x - s), \tag{13}$$

but due to the fact that this purely-physical consideration has no effect on the underlying mathematics, this point of view is generally overlooked. Several other notations are also known to exist for a Green's function, some of which include the use of a lower-case $g = g(x, s)$ in place of $G(x, s)$ (Stakgold 1979) as well as the inclusion of a vertical line instead of a comma, e.g., $G(x, s) = G(x \mid s)$ (Duffy 2001).

In other instances, literature presents definitions which are intimately connected to the contexts in which they're presented. For example, some authors define Green's functions to be functions which satisfy a certain set of conditions, e.g.,existence on a special kind of domain, association with a very particular differential operator $\mathcal{L}$, or satisfaction of a precise set of boundary conditions. One of the most common such examples can be found in notes by, e.g., Speck, where a Green's function is defined to satisfy $\Delta_s G(x,s)=\delta(x)$ for points $(x,s)\in\Omega\times\Omega$ and $G(x,\sigma)=0$ for all points σ lying in the boundary $\partial\Omega$ of Ω (Speck 2011). This particular definition presents an integral kernel corresponding to the solution of a generalized Poisson's equation and would therefore face obvious limitations when being adapted to a more general setting. On the other hand, such examples aren't without their benefits. In the case of the generalized Poisson example above, for instance, each such Green's function G can be split so that

$$G(x,s)=g_f(x,s)+u_R(x,s) \tag{14}$$

where $-\Delta g_f(x,s)=\delta(x-s)$ and $-\Delta u_R(x,s)=0$ for the regular Laplacian $\Delta=\Delta_s$ (Hartman 2013). In such situations, $g_f=g_f(x,s)$ is known as the fundamental solution of the underlying differential equation and $u_R=u_R(x,s)$ is known as its regular solution; as such, g_f and u_R are sometimes called the fundamental and regular parts of G, respectively.

Several fundamental properties of a general Green's function follow immediately (or almost so) from its definition and carry over to all particular instances. For example, if the kernel of the operator $\mathcal{L}$ is non-trivial, then there may be several Green's functions associated to a single operator; as a result, one must exhibit caution when referring to "the" Green's function. Green's functions satisfy an adjoint symmetry in their two arguments so that

$$G(x,s)=G^*(s,x) \tag{15}$$

where here, G^* is defined to be the solution of the equation

$$\mathcal{L}^* G^*(s,x)=\delta(x-s). \tag{16}$$

Here, $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$. One immediate corollary of this fact is that for self-adjoint operators $\mathcal{L}$, G is symmetric:

$$G(x, s) = G(s, x). \quad (17)$$

This identity is often called the reciprocity principle and says, in physical terms, that the response at x caused by a unit source at s is the same as the response at s due to a unit force at x (Stakgold 1979).

The essential property of any Green's function is that it provides a way to describe the response of an arbitrary differential equation solution to some kind of source term in the presence of some number of boundary conditions. Some authors consider a Green's function to serve roughly an analogous role in the theory of partial differential equations as do Fourier series in the solution of ordinary differential equations .

For more abstract scenarios, a number of concepts exist which serve as context-specific analogues to the notion of a Green's function. For instance, in functional analysis, it is often useful to consider a so-called generalized Green's function which has many analogous properties when integrated abstractly against functionals rather than functions. Indeed, such generalizations have yielded an entirely analogous branch of theoretical PDE analysis and are themselves the focus of a large amount of research.

Check In Progress-I

Q. 1 Define Green's Function.

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

Q.2 Write Definition of Green's Function.

Solution : . . . .
. . . .
. . . .

. . . .

. . . .

. . . .

12.3.1 Green's Function--Poisson's Equation

Poisson's equation is

$$\nabla^2 \phi = 4\pi\rho, \tag{1}$$

where ϕ is often called a potential function and ρ a density function, so the differential operator in this case is $\tilde{L} = \nabla^2$. As usual, we are looking for a Green's function $G(\mathbf{r}_1, \mathbf{r}_2)$ such that

$$\nabla^2 G(\mathbf{r}_1, \mathbf{r}_2) = \delta^3(\mathbf{r}_1 - \mathbf{r}_2). \tag{2}$$

But from Laplacian,

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta^3(\mathbf{r} - \mathbf{r}'), \tag{3}$$

so

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}, \tag{4}$$

and the solution is

$$\phi(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')[4\pi\rho(\mathbf{r}')]\, d^3\mathbf{r}' = -\int \frac{\rho(\mathbf{r}')\, d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \tag{5}$$

Expanding $G(\mathbf{r}_1, \mathbf{r}_2)$ in the spherical harmonics Y_l^m gives

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_l^m(\theta_1, \phi_1) \bar{Y}_l^m(\theta_2, \phi_2), \tag{6}$$

where $r_<$ and $r_>$ are greater than/less than symbols. this expression simplifies to

$$g(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} p_l(\cos\gamma), \tag{7}$$

Notes

where P_l are Legendre polynomials, and $\cos\gamma \equiv \mathbf{r}_1 \cdot \mathbf{r}_2$. Equations (6) and (7) give the addition theorem for Legendre polynomials.

In cylindrical coordinates, the Green's function is much more complicated,

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi^2} \sum_{m=-\infty}^{\infty} \int_0^{\infty} I_m(k\rho_<) K_m(k\rho_>) e^{im(\phi_1-\phi_2)} \cos[k(z_1-z_2)]\, dk, \quad (8)$$

where $I_m(x)$ and $K_m(x)$ are modified Bessel functions of the first and second kinds.

12.3.2 Green's Function--Helmholtz Differential Equation

The inhomogeneous Helmholtz differential equation is

$$\nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = \rho(\mathbf{r}), \quad (1)$$

where the Helmholtz operator is defined as $\tilde{L} \equiv \nabla^2 + k^2$. The Green's function is then defined by

$$\left(\nabla^2 + k^2\right) G(\mathbf{r}_1, \mathbf{r}_2) = \delta^3(\mathbf{r}_1 - \mathbf{r}_2). \quad (2)$$

Define the basis functions ϕ_n as the solutions to the homogeneous Helmholtz differential equation

$$\nabla^2\phi_n(\mathbf{r}) + k_n^2\phi_n(\mathbf{r}) = 0. \quad (3)$$

The Green's function can then be expanded in terms of the ϕ_ns,

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1), \quad (4)$$

and the delta function as

$$\delta^3(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{n=0}^{\infty} \phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2). \quad (5)$$

Plugging (◇) and (◇) into (◇) gives

$$\nabla^2\left[\sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1)\right] + k^2\sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1) = \sum_{n=0}^{\infty}\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2). \tag{6}$$

Using (◇) gives

$$-\sum_{n=0}^{\infty} a_n(\mathbf{r}_2)k_n^2\phi_n(\mathbf{r}_1) + k^2\sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1) = \sum_{n=0}^{\infty}\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2) \tag{7}$$

$$\sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1)\left(k^2 - k_n^2\right) = \sum_{n=0}^{\infty}\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2). \tag{8}$$

This equation must hold true for each n, so

$$a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1)\left(k^2 - k_n^2\right) = \phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2) \tag{9}$$

$$a_n(\mathbf{r}_2) = \frac{\phi_n(\mathbf{r}_2)}{k^2 - k_n^2}, \tag{10}$$

and (◇) can be written

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n=0}^{\infty}\frac{\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2)}{k^2 - k_n^2}. \tag{11}$$

The general solution to (◇) is therefore

$$\psi(\mathbf{r}_1) = \int G(\mathbf{r}_1, \mathbf{r}_2)\rho(\mathbf{r}_2)\,d^3\mathbf{r}_2 \tag{12}$$

$$= \sum_{n=0}^{\infty}\int\frac{\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2)\rho(\mathbf{r}_2)}{k^2 - k_n^2}\,d^3\mathbf{r}_2. \tag{13}$$

12.3.3 Green Function for Ordinary Differential Equations

Let L be the differential operator generated by the differential polynomial

$$l[y] = \sum_{k=0}^{n} p_k(x)\frac{d^k y}{dx^k}, \qquad a < x < b,$$

and the boundary conditions $U_j[y] = 0$, $j = 1, \ldots, n$, where

$$U_j[y] = \sum_{k=0}^{n} \alpha_{jk}\, y^{(k)}(a) + \beta_{jk}\, y^{(k)}(b).$$

The Green function of L is the function $G(x, \xi)$ that satisfies the following conditions:

Notes

1) $G(x, \xi)$ is continuous and has continuous derivatives with respect to x up to order $n-2$ for all values of x and ξ in the interval $[a, b]$.

2) For any given ξ in (a, b) the function $G(x, \xi)$ has uniformly-continuous derivatives of order n with respect to x in each of the half-intervals $[a, \xi)$ and $(\xi, b]$ and the derivative of order $n-1$ satisfies the condition

$$\frac{\partial^{n-1}}{\partial x^{n-1}} G(\xi+, \xi) - \frac{\partial^{n-1}}{\partial x^{n-1}} G(\xi-, \xi) = \frac{1}{p_n(\xi)}$$

if $x = \xi$.

3) In each of the half-intervals $[a, \xi)$ and $(\xi, b]$ the function $G(x, \xi)$, regarded as a function of x, satisfies the equation $l[G] = 0$ and the boundary conditions $U_j[G] = 0$, $j = 1, \dots, n$.

If the boundary value problem $Ly = 0$ has trivial solutions only, then L has one and only one Green function [1]. For any continuous function f on $[a, b]$ there exists a solution of the boundary value problem $Ly = f$, and it can be expressed by the formula

$$y(x) = \int_a^b G(x, \xi) f(\xi)\, d\xi \, .$$

If the operator L has a Green function $G(x, \xi)$, then the adjoint operator L^* also has a Green function, equal to $\overline{G(\xi, x)}$. In particular, if L is self-adjoint ($L = L^*$), then $G(x, \xi) = \overline{G(\xi, x)}$, i.e. the Green function is a Hermitian kernel in this case. Thus, the Green function of the self-adjoint second-order operator L generated by the differential operator with real coefficients

$$l[y] = \frac{d}{dx}\left(p\frac{dy}{dx}\right) + q(x)y, \qquad a < x < b,$$

and the boundary conditions $y(a) = 0$, $y(b) = 0$ has the form:

$$G(x, \xi) = \begin{cases} Cy_1(x)y_2(\xi) & \text{if } x \le \xi, \\ Cy_1(\xi)y_2(x) & \text{if } x > \xi \, . \end{cases}$$

Here $y_1(x)$ and $y_2(x)$ are arbitrary independent solutions of the equation $l[y]=0$ satisfying, respectively, the conditions $y_1(a)=0$, $y_2(b)=0$; $C=[p(\xi)W(\xi)]^{-1}$, where W is the Wronski determinant (Wronskian) of y_1 and y_2. It can be shown that C is independent of ξ.

If the operator L has a Green function, then the boundary eigen value problem $Ly=\lambda y$ is equivalent to the integral equation $y(x)=\lambda\int_a^b G(x,\xi)y(\xi)\,d\xi$, to which Fredholm's theory is applicable (cf. also Fredholm theorems). For this reason the boundary value problem $Ly=\lambda y$ can have at most a countable number of eigen values λ_1, λ_2 ,..., without finite limit points. The conjugate problem has complex-conjugate eigen values of the same multiplicity. For each λ that is not an eigen value of L it is possible to construct the Green function $G(x,\xi,\lambda)$ of the operator $L-\lambda I$, where I is the identity operator. The function $G(x,\xi,\lambda)$ is a meromorphic function of the parameter λ; its poles can be eigen values of L only. If the multiplicity of the eigen value λ_0 is one, then

$$G(x,\xi,\lambda)=\frac{u_0(x)\overline{v_0(\xi)}}{\lambda-\lambda_0}+G_1(x,\xi,\lambda),$$

where $G_1(x,\xi,\lambda)$ is regular in a neighbourhood of the point λ_0, and $u_0(x)$ and $v_0(x)$ are the eigen functions of L and L^* corresponding to the eigen values λ_0 and $\overline{\lambda_0}$ and normalized so that

$$\int_a^b u_0(x)\overline{v_0(x)}\,dx=1.$$

If $G(x,\xi,\lambda)$ has infinitely-many poles and if these are of the first order only, then there exists a complete biorthogonal system

$$u_1(x), u_2(x), \ldots; \qquad v_1(x), v_2(x), \ldots,$$

of eigen functions of L and L^*. If the eigen values are numbered in increasing sequence of their absolute values, then the integral

Notes

$$I_R(x, f) = \frac{1}{2\pi i} \int_{|\lambda|=R} d\lambda \int_a^b G(x, \xi, \lambda) f(\xi)\, d\xi$$

is equal to the partial sum

$$S_k(x, f) = \sum_{|\lambda_n|<R} u_n(x) \int_a^b f(\xi) \overline{v_n(\xi)}\, d\xi$$

of the expansion of f with respect to the eigen functions of L. The positive number R is so selected that the function $G(x, \xi, \lambda)$ is regular in λ on the circle $|\lambda| = R$. For a regular boundary value problem and for any piecewise-smooth function f in the interval $a < x < b$, the equation

$$\lim_{R\to\infty} I_R(x, f) = \frac{1}{2}[f(x+0) + f(x-0)]$$

is valid, that is, an expansion into a convergent series is possible [1].

If the Green function $G(x, \xi, \lambda)$ of the operator $L - \lambda I$ has multiple poles, then its principal part is expressed by canonical systems of eigen and adjoint functions of the operators L and L^* [2].

In the case considered above, the boundary value problem $Ly = 0$ has no non-trivial solutions. If, on the other hand, such non-trivial solutions exist, a so-called generalized Green function is introduced. Let there exist, e.g., exactly m linearly independent solutions of the problem $Ly = 0$. Then a generalized Green function $\tilde{G}(x, \xi)$ exists that has properties 1) and 2) of an ordinary Green function, satisfies the boundary conditions as a function of x if $a < \xi < b$ and, in addition, is a solution of the equation

$$l_x[y] = -\sum_{k=1}^{m} \phi_k(x) \overline{v_k(\xi)}.$$

Here $\{v_k(x)\}_{k=1}^{m}$ is a system of linearly independent solutions of the adjoint problem $L^* y = 0$, while $\{\phi_k(x)\}_{k=1}^{m}$ is an arbitrary system of continuous functions biorthogonal to it. Then

$$y(x) = \int_a^b \widetilde{G}(x, \xi) f(\xi) d\xi$$

is the solution of the boundary value problem $Ly = f$ if the function f is continuous and satisfies the solvability criterion, i.e. is orthogonal to all v_k.

If $\widetilde{G}_0$ is one of the generalized Green functions of L, then any other generalized Green function can be represented in the form

$$\widetilde{G}(x, \xi) = \widetilde{G}_0(x, \xi) + \sum_{k=1}^{m} u_k(x) \psi_k(\xi),$$

where $\{u_k(x)\}$ is a complete system of linearly independent solutions of the problem $Ly = 0$, and $\psi_k(\xi)$ are arbitrary continuous functions.

12.3.4 Green function for partial differential equations

1) Elliptic equations. Let A be the elliptic differential operator of order m generated by the differential polynomial

$$a(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$$

in a bounded domain $\Omega \subset \mathbf{R}^N$ and the homogeneous boundary conditions $B_j u = 0$, where B_j are boundary operators with coefficients defined on the boundary $\partial \Omega$ of Ω, which is assumed to be sufficiently smooth. A function $G(x, y)$ is said to be a Green function for A if, for any fixed $y \in \Omega$, it satisfies the homogeneous boundary conditions $B_j G(x, y) = 0$ and if, regarded as a generalized function, it satisfies the equation

$$a(x, D) G(x, y) = \delta(x - y) .$$

In the case of operators with smooth coefficients and normal boundary conditions, which ensure that the solution of the homogeneous boundary value problem is unique, a Green function exists and the solution of the boundary value problem $Au = f$ can be represented in the form (cf. [4])

$$u(x) = \int_{\Omega} G(x, y) f(y)\, dy .$$

In such a case the uniform estimates for $x, y \in \overline{\Omega}$,

$$|G(x, y)| \leq C|x - y|^{m-n} \quad \text{if } m < n,$$
$$|G(x, y)| \leq C + C|\ln |x - y|| \quad \text{if } m = n,$$

are valid for the Green function, and the latter is uniformly bounded if $m > n$.

The boundary eigen value problem $Au = \lambda u$ is equivalent to the integral equation

$$u(x) = \lambda \int_{\Omega} G(x, y) u(y)\, dy,$$

to which Fredholm's theory (cf. [5]) is applicable (cf. Fredholm theorems). Here, the Green function of the adjoint boundary value problem is $\overline{G(y, x)}$. It follows, in particular, that the number of eigen values is at most countable, and there are no finite limit points; the adjoint boundary value problem has complex-conjugate eigen values of the same multiplicity.

A Green function has been more thoroughly studied for second-order equations, since the nature of the singularity of the fundamental solution can be explicitly written out. Thus, for the Laplace operator the Green function has the form

$$G(x, y) = -\frac{\Gamma(n/2)}{2\pi^{n/2}(n-2)}|x - y|^{2-n} + \gamma(x, y) \quad \text{if } n > 2,$$
$$G(x, y) = +\frac{1}{2\pi}\ln|x - y| + \gamma(x, y) \quad \text{if } n = 2,$$

where $\gamma(x, y)$ is a harmonic function in Ω chosen so that the Green function satisfies the boundary condition.

The Green function $G(x, y)$ of the first boundary value problem for a second-order elliptic operator $\alpha(x, D)$ with smooth coefficients in a domain Ω with Lyapunov-type boundary $\partial\Omega$, makes it possible to express the solution of the problem

$$\alpha(x, D)u(x) = f(x) \quad \text{if} \quad x \in \Omega, \quad u|_{\partial\Omega} = \phi,$$

in the form

$$u(x) = \int\limits_{\Omega} G(x, y) f(y)\, dy + \int\limits_{\partial\Omega} \frac{\partial}{\partial \nu_y} G(x, y)\, \phi(y)\, d\sigma_y ,$$

where $\partial / \partial \nu_y$ is the derivative along the outward co-normal of the operator $a(x, D)$ and $d\sigma_y$ is the surface element on $\partial\Omega$.

If the homogeneous boundary condition $Au = 0$ has non-trivial solutions, a generalized Green function is introduced, just as for ordinary differential equations. Thus, a generalized Green function, the so-called Neumann function [3], is available for the Laplace operator.

2) Parabolic equations. Let P be the parabolic differential operator of order m generated by the differential polynomial

$$p\left(x, t, D_X, \frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} - \sum_{|\alpha| \le m} a_\alpha(x, t) D_X^\alpha ,$$

$$x \in \Omega , \qquad t > 0,$$

and the homogeneous initial and boundary conditions

$$u(x, 0) = 0, \qquad B_j u(x, t) = 0,$$

where B_j are boundary operators with coefficients defined for $x \in \partial\Omega$ and $t \geq 0$. The Green function of the operator P is a function $G(x, t, y, \tau)$ which for arbitrary fixed (y, τ) with $t > \tau \geq 0$ and $y \in \Omega$ satisfies the homogeneous boundary conditions $B_j = 0$ and also satisfies the equation

$$p\left(x, t, D_X, \frac{\partial}{\partial t}\right) G(x, t, y, \tau) = \delta(x - y, t - \tau) .$$

For operators with smooth coefficients and normal boundary conditions, which ensures the uniqueness of the solution of the problem $pu = 0$, a Green function exists, and the solution of the equation

$$p\left(x, t, D_X, \frac{\partial}{\partial t}\right) u(x, t) = f(x, t)$$

satisfying the homogeneous boundary conditions and the initial conditions $u(x, 0) = \phi(x)$, has the form

$$u(x,t)=\int_0^t d\tau\int_\Omega G(x,t,y,\tau)f(y,\tau)\,dy+$$
$$+\int_\Omega G(x,t,y,0)\,\phi(y)\,dy.$$

In the study of elliptic or parabolic systems the Green function is replaced by the concept of a Green matrix, by means of which solutions of boundary value problems with homogeneous boundary conditions for these systems are expressed as integrals of the products of a Green matrix by the vectors of the right-hand sides and the initial conditions [7].

Green functions are named after G. Green (1828), who was the first to study a special case of such functions in his studies on potential theory.

12.3.5 Green Function in Function Theory

In the theory of functions of a complex variable, a (real) Green function is understood to mean a Green function for the first boundary value problem for the Laplace operator, i.e. a function of the type

$$G(z,z_0)=\ln\frac{1}{|z-z_0|}+\gamma(z,z_0),\qquad z\in\Omega,$$

where $z=x+iy$ is the complex variable, $z_0=x_0+iy_0$ is the pole of the Green function, $z_0\in\Omega$, and $\gamma(z,z_0)$ is a harmonic function of z which takes the values $-\ln 1/|z-z_0|$ at the boundary $\partial\Omega$. Let the domain Ω be simply-connected and let $w=f(z,z_0)$ be the analytic function which realizes the conformal mapping of Ω onto the unit disc so that z_0 maps to the centre of the disc, and such that $f(z_0,z_0)=0$, $f'(z_0,z_0)>0$.

Then

$$G(z,z_0)=\ln\frac{1}{|f(z,z_0)|}.$$

If $H(z,z_0)$ is the harmonic function conjugate with $G(z,z_0)$, $H(z_0,z_0)=0$, then the analytic function $F(z,z_0)=G(z,z_0)+iH(z,z_0)$ is said to be a complex Green function of Ω with pole z_0. The inversion of formula (2) yields

$f(z, z_0) = e^{-F(z, z_0)}$.

Formulas (2) and (3) show that the problems of constructing a conformal mapping of Ω into the disc and of finding a Green function are equivalent. The Green functions $G(z, z_0)$, $F(z, z_0)$ are invariant under conformal mappings, which may sometimes facilitate their identification (see Mapping method).

In the theory of Riemann surfaces it is more convenient to define Green functions with the aid of a minimum property, valid for a function (1): Of all functions $U(z, z_0)$ on a Riemann surface Ω that are positive and harmonic for $z \neq z_0$ and have in a neighbourhood of z_0 the form

$$U(z, z_0) = \ln \frac{1}{|z - z_0|} + \gamma(z, z_0),$$

where $\gamma(z, z_0)$ is a harmonic function which is regular on the entire surface Ω , the Green function, if it exists, is the least, i.e. $G(z, z_0) \leq U(z, z_0)$. Here, the existence of a Green function is typical for Riemann surfaces of hyperbolic type. If a Green function is thus defined, it no longer vanishes, generally speaking, anywhere on the (ideal) boundary of the Riemann surface. The situation is similar in potential theory (see also Potential theory, abstract). For an arbitrary open set Ω, e.g. in the Euclidean space $\mathbf{R}^n$, $n \geq 2$, the Green function $G(x, x_0)$ can also be defined with the aid of the minimum property discussed above, but for $n \geq 3$ the expression $|x - x_0|^{2-n}$ should be substituted for $\ln 1/|x - x_0|$ in formula (4). In general, such a Green function does not necessarily tend to zero as the boundary $\partial \Omega$ is approached. A Green function does not exist for Riemann surfaces of parabolic type or for certain domains in $\mathbf{R}^2$ (e.g. for $\Omega = \mathbf{R}^2$).

Check In Progress-II

Q. 1 Define Green Function for Ordinary Differential Equations.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

Q.2 Define Green function for partial differential equations.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

12.4 FREDHOLM THEOREMS

Theorem 1.The homogeneous equation

$$\phi(x) - \lambda \int_a^b K(x,s)\,\phi(s)\,ds = 0$$

and its transposed equation

$$\psi(x) - \lambda \int_a^b K(s,x)\,\psi(s)\,ds = 0$$

have, for a fixed value of the parameter λ, either only the trivial solution, or have the same finite number of linearly independent solutions: $\phi_1, \dots, \phi_n; \psi_1, \dots, \psi_n$.

Theorem 2. For a solution of the inhomogeneous equation

$$\phi(x) - \lambda \int_a^b K(x,s)\,\phi(s)\,ds = f(x)$$

to exist it is necessary and sufficient that its right-hand side be orthogonal to a complete system of linearly independent solutions of the corresponding homogeneous transposed equation (2):

$$\int_a^b f(x)\,\psi_j(x)\,dx=0, \qquad j=1,\dots,n.$$

Theorem 3. (the Fredholm alternative). Either the inhomogeneous equation (3) has a solution, whatever its right-hand side f, or the corresponding homogeneous equation (1) has non-trivial solutions.

Theorem 4. The set of characteristic numbers of equation (1) is at most countable, with a single possible limit point at infinity.

For the Fredholm theorems to hold in the function space $L_2\,[a,b]$ it is sufficient that the kernel K of equation (3) be square-integrable on the set $[a,b]\times[a,b]$ (a and b may be infinite). When this condition is violated, (3) may turn out to be a non-Fredholm integral equation. When the parameter λ and the functions involved in (3) take complex values, then instead of the transposed equation (2) one often considers the adjoint equation to (1):

$$\psi(x)-\overline{\lambda}\int_a^b \overline{K(s,x)}\,\psi(s)\,ds=0.$$

In this case condition (4) is replaced by

$$\int_a^b f(x)\,\overline{\psi_j(x)}\,dx=0, \qquad j=1,\dots,n.$$

These theorems were proved by E.I. Fredholm

12.5DIFFERENTIAL OPERATOR

A generalization of the concept of a differentiation operator. A differential operator (which is generally discontinuous, unbounded and non-linear on its domain) is an operator defined by some differential expression, and acting on a space of (usually vector-valued) functions (or sections of a differentiable vector bundle) on differentiable manifolds or else on a space dual to a space of this type. A differential expression is a mapping λ of a set Ω in the space of sections of a vector bundle ξ with base M into the space of sections of a vector bundle η with the same base such that for any point $p \in M$ and arbitrary sections $f, g \in \Omega$ the coincidence of their k -jets (cf. Jet) at p entails the coincidence of λf and λg at that point. The smallest number k which meets this condition for all $p \in M$ is said to be the order of the differential expression and the order of the differential operator defined by this expression.

A differential operator on a manifold M without boundary often proves to be an extension of an operator which is defined in a natural manner by a fixed differential expression on some set, open in an appropriate topology, of infinitely (or sufficiently often) differentiable sections of a given vector bundle ξ with base M, and thus permits a natural extension to the case of sheaves of germs of sections of differentiable vector bundles. A differential operator L on a manifold M with boundary ∂M is often defined as an extension of an analogous operator which is naturally defined by a differential expression on the set of differentiable functions (or sections of a vector bundle), the restrictions of which to ∂M lie in the kernel of some differential operator l on ∂M (or satisfies some other conditions definable by some requirements to be satisfied in the domain of values of an operator l on the restrictions of the functions from the domain of definition of L, such as inequalities); the differential operator l is said to define the boundary conditions for the differential operator L. Linear differential operators on spaces dual to spaces of functions (or sections) are defined as operators dual to the differential operators of the above type on these spaces.

Examples.

1) Let F be a real-valued function of $k+2$ variables $x, y_0, \ldots, y_k$, defined in some rectangle $\Delta = I \times J_0 \times \ldots \times J_k$; the differential expression

$$Du = F\left(x, u, \frac{du}{dx}, \ldots, \frac{d^k u}{dx^k}\right)$$

(where F usually satisfies some regularity conditions such as measurability, continuity, differentiability, etc.) defines a differential operator D on the manifold I, the domain of definition Ω of which consists of all functions $u \in C^k(I)$ satisfying the condition $u^{(i)}(x) \in J_i$ for $i = 1, 2, \ldots$. If F is continuous, D may be considered as an operator on $C(I)$ with domain of definition Ω; the differential operator D is said to be a general ordinary differential operator. If F depends on y_k, the order of D is k. D is said to be quasi-linear if it depends linearly on y_k; it is linear if F depends linearly on $y_0, \ldots, y_k$; it is said to be linear with constant coefficients if F is independent of x and if D is a linear differential operator. The remaining differential operators are said to be non-linear. If certain conditions as to the regularity of F are satisfied, a quasi-linear operator may be extended to a differential operator from one Sobolev space into another.

2) Let $x = (x^1, \ldots, x^N)$ run through a domain $\mathcal{O}$ in $\mathbf{R}^N$, let $F = (x, u, D^{(n)}(u))$ be a differential expression defined by a real-valued function F on the product of $\mathcal{O}$ and some open rectangle ω, where $D^{(n)}(u)$ is a set of partial derivatives of the type $D^\alpha u = \partial^{\alpha_1 + \ldots + \alpha_N} u / (\partial x^1)^{\alpha_1} \ldots (\partial x^N)^{\alpha_N}$, where $\alpha_1 + \ldots + \alpha_N \leq n$, and, as in example 1), let the function F satisfy certain regularity conditions. The differential operator defined by this expression on the space of sufficiently often differentiable functions on $\mathcal{O}$ is known as a general partial differential operator. As in example 1), one defines non-linear, quasi-linear and linear partial differential operators and the order of a partial differential operator; a differential operator is said to be elliptic, hyperbolic or parabolic if it is defined by a differential expression of the respective type. One sometimes considers functions F depending on derivatives of all orders (e.g. as their formal linear combination); such differential expressions, although not defining a differential operator in the ordinary sense, can nevertheless be brought into correspondence with certain operators (e.g. on spaces of germs of analytic functions), and are known as differential operators of infinite order.

3) The previous examples may be extended to include the complex-valued case or the case of functions with values in a locally compact,

totally disconnected field and (at least in the case of linear differential operators) even to a more general situation (cf. Differential algebra).

4) Systems of differential expressions define differential operators on spaces of vector functions. For example, the Cauchy–Riemann differential operator, defined by the expression $\{\partial u/\partial x-\partial v/\partial y,\ \square\,\partial u/\partial y+\partial v/\partial x\}$, converts the space of pairs of harmonic functions on the plane into itself.

In the definition of a differential operator and of its generalizations one often employs (besides ordinary derivatives) generalized derivatives, which appear in a natural manner when considering extensions of differential operators defined on differentiable functions, and weak derivatives, related to the transition to the adjoint operator. Moreover, derivatives of fractional and negative orders appear when the differentiation is defined by means of a Fourier transform (or some other integral transform), applicable to the domain of definition and range of such a generalized differential operator (cf. Pseudo-differential operator). This is done in order to obtain the simplest possible representation of the corresponding differential operator of a function F and to attain a reasonable generality in the formulation of problems and satisfactory properties of the objects considered. In this way, a functional or operational calculus is obtained, extending the correspondence between the differentiation operator and the operator of multiplication by the independent variable as realized in the Fourier transform.

Problems in the theory of differential equations — such as problems of existence, uniqueness, regularity, continuous dependence of the solutions on the initial data or on the right-hand side, the explicit form of a solution of a differential equation defined by a given differential expression — are readily interpreted in the theory of operators as problems on the corresponding differential operator defined on suitable function spaces — viz. as problems on kernels, images, the structure of the domain of definition of a given differential operator L or of its extension, continuity of the inverse of the given differential operator and explicit construction of this inverse operator. Problems of the approximation of solutions and of the construction of approximate solutions of differential equations are also readily generalized and improved as problems on the corresponding differential operators, viz. — selection of natural topologies in the domain of definition and in the range such that the operator L (if the solutions are unique) realizes a homeomorphism of the domains of definition and ranges in these topologies (this theory is connected with the theory of interpolation and scales (grading) of function spaces, in particular for linear and quasi-linear differential operators). Another example is the selection of differential operators close to a given operator in some definite sense (which makes it possible by using appropriate topologies in the space of differential operators, to justify methods of approximation of equations, such as the regularization and the penalty method, and iterated regularization methods). The theory of differential operators makes it possible to apply classical methods in the theory of operators, e.g. the theory of compact operators, and the method of contraction mappings in various existence and uniqueness theorems for differential equations, in the theory of bifurcation of solutions and in non-linear eigen value problems. Other applications utilize a natural order structure present in function spaces on which a differential operator is defined (in particular, the theory of monotone operators), or use methods of linear analysis (the theory of duality, convex sets, dual or dissipative operators). Again, variational methods and the theory of extremal problems or the presence of certain supplementary structures (e.g. complex, symplectic, etc.) can be used in order to clarify the structure of the kernel and range of the differential operator, i.e. to obtain information on the solution space of the respective equations. Many problems connected with differential expressions necessitate a study of differential inequalities, which are closely connected with multi-valued differential operators.

Thus, the theory of differential operators makes it possible to eliminate a number of difficulties involved in the classical theory of differential equations. The utilization of various extensions of classical differential operators leads to the concept of generalized solutions of the corresponding differential equations (which necessarily proved to be classical in several cases connected with, say, elliptic problems), while the utilization of the linear structure makes it possible to introduce the concept of weak solutions of differential equations. In choosing a suitable extension of a differential operator as defined by a differential expression, a priori estimates of solutions connected with such an expression are of importance, since they permit one to identify function spaces on which the extended operator is continuous or bounded.

Moreover, the theory of differential operators also makes it possible to formulate and solve many new problems, which are qualitatively different from the classical problems in the theory of differential equations. Thus, in the study of non-linear operators it is of interest to study the structure of the set of its stationary points and the action of the operator in a neighbourhood of them, as well as the classification of these singular points, and the stability of the type of the singular point when the respective differential operator is perturbed. Other subjects of interest in the theory of linear differential operators are the description and the study of the spectrum of a differential operator, the calculation of its index, the structure of invariant subspaces of the differential operator, the harmonic analysis of a given differential operator (in particular, the decomposition, which requires a preliminary study of the completeness of the system of eigen functions and associated functions). There is also the study of linear and non-linear perturbations of a given differential operator. These results are of special interest for elliptic differential operators generated by symmetric differential expressions in the context of the theory of self-adjoint operators on a Hilbert space (in particular, in the spectral theory of these operators and the theory of extensions of symmetric operators). The theory of various hyperbolic and parabolic (not necessarily linear) differential operators is connected with the theory of groups and semi-groups of operators on locally convex spaces.

Next to the linear class of differential operators, perhaps the most intensively studied class are differential operators which are either invariant or which vary according to a specific law when certain transformations constituting a group (or a semi-group) G are acting in their domain of definition, and hence also on the differential expression. These include, for instance, invariant differential operators connected with the representations of a group G; the covariant derivative or, more generally, differential operators on spaces of differentiable tensor fields, where G is the group of all diffeomorphisms (the so-called atomization);

many examples of operators in theoretical physics, etc. Such functional-geometric methods are also useful in the study of differential operators with so-called hidden symmetry (see, for example, Korteweg–de Vries equation).

The theory of differential operators as part of the general theory of operators has lately been of increasing importance not merely in the theory of differential equations, but in modern analysis in general. It yields not only important specific examples of unbounded operators (particularly in the theory of linear differential operators), but also tools for the representation and means of study of other objects of various natures. For instance, any generalized function (and even a hyperfunction) is locally obtained by the action of a certain generalized differential operator on a continuous function. Finally, differential operators are becoming more important in other branches of mathematics and effect these to an increasing extend. E.g., one solution of the so-called index problem (cf. Index formulas) connects the topological characteristics of a manifold with the presence of a particular class of differential operators on it; from this it is possible to deduce the properties of elliptic complexes on this manifold.

12.6 LET'S SUM UP

1. We study Green's function of
   an inhomogeneous linear differential operator defined on a domain with specified initial conditions or boundary conditions is its impulse response.

This means that if *L* is the linear differential operator, then

- the Green's function *G* is the solution of the equation $LG = \delta$, where δ is Dirac's delta function;
- the solution of the initial-value problem $Ly = f$ is the convolution $(G * f)$, where *G* is the Green's function.

2. We also study Poisson's equation is

$$\nabla^2 \phi = 4\pi\rho,$$

where ϕ is often called a potential function and ρ a density function

3. We learnt The inhomogeneous Helmholtz differential equation is

$$\nabla^2 \psi(\mathbf{r}) + k^2 \psi(\mathbf{r}) = \rho(\mathbf{r}),$$

where the Helmholtz operator is defined as $\tilde{L} \equiv \nabla^2 + k^2$.

12.7 KEYWORD

Differential Operator : A *differential operator* (which is generally discontinuous, unbounded and non-linear on its domain) is an *operator defined* by some *differential* expression, and acting on a space of (usually vector-valued) functions (or sections of a differentiable vector bundle) on differentiable manifolds.

Hermitian kernel : A Hermitian kernel is a kernel that satisfies the property. K. ∗(x,t) = K(t,x) = K(x,t) in the square Q(a,b) = {(x,t): a ≤ x ≤ b and a ≤ t ≤ b}. We assume as usual that K(x,t) is continuous in Q(a,b).

Helmholtz Differential : Helmholtz, is the linear partial differential equation. where is the Laplacian, is the wave number, and is the amplitude. This is also an eigenvalue equation.

12.8 QUESTIONS FOR REVIEW

Q. 1 State Green's Function.

Q. 2 Define Green's Function with Poison's Function

Q. 3 Define Green's Function--Helmholtz Differential Equation.

Q. 4 STATE FREDHOLM THEOREMS IN THREE CASE.

Q. 5 Define Differential Operator.

12.9 SUGGESTION READING AND REFERENCES

- M.A. Naimark, "Lineare Differentialoperatoren" , Akademie

Verlag (1960) (Translated from Russian) MR0216049

- M.V. Keldysh, "On the characteristic values and characteristic functions of certain classes of non-self-adjoint equations" *Dokl. Akad. Nauk. SSSR* , **77** : 1 (1951) pp. 11–14 (In Russian)

- V.V. Sobolev, "Course in theoretical astrophysics" , NASA , Washington, D.C. (1969) (Translated from Russian)

- L. Bers, F. John, M. Schechter, "Partial differential equations" , Interscience (1964) MR0163043 Zbl 0126.00207

- L. Gårding, "Dirichlet's problem for linear elliptic partial differential equations" *Math. Scand.* , **1** : 1 (1953) pp. 55–72 MR64979

- A. Friedman, "Partial differential equations of parabolic type" , Prentice-Hall (1964) MR0181836 Zbl 0144.34903

- S.D. Eidel'man, "Parabolic systems" , North-Holland (1969) (Translated from Russian) Zbl 0181.37403

12.10 ANSWER TO CHECK IN PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 4

Q. 2 Check in Section 3

Check In progress-II

Answer Q. 1 Check in Section 4.4

Q. 2 Check in Section 4.5

UNIT 13: STURM COMPARISON THEOREMS AND OSCILLATIONS

STRUCTURE

13.0 OBJECTIVES

- In this unit we study Sturm-Picone Comparison Theorem
- We also study second-order linear equation: $x^{\Delta\Delta}(t)+a_1(t)x^{\Delta\sigma}(t)+a_2(t)x^{\sigma}(t)=0$
- We study Sturm-Liouville Equation and its proof
- We also study An nth order homogeneous linear differential operator (equation)

  $Ly \equiv y^{(n)}+p_1(x)y^{(n-1)}+\ldots+p_n(x)y=0$

13.1 INTRODUCTION

we consider the following second-order linear equations:

$$(p_1(t)x^{\Delta}(t))^{\Delta}+q_1(t)x^{\sigma}(t)=0, \quad (1.1)$$

$$(p_2(t)y^{\Delta}(t))^{\Delta} + q_2(t)y^{\sigma}(t) = 0, \qquad (1.2)$$

where $t \in [\alpha,\beta] \cap \mathbf{T}$, $p_1^{\Delta}(t)$, $p_2^{\Delta}(t)$, $q_1(t)$, and $q_2(t)$ are real and rd-continuous functions in $[\alpha,\beta] \cap \mathbf{T}$. Let $\mathbf{T}$ be a time scale, $\sigma(t)$ be the forward jump operator in $\mathbf{T}$, y^{Δ} be the delta derivative, and $y^{\sigma}(t) := y(\sigma(t))$.

First we briefly recall some existing results about differential and difference equations. As we well know, in 1909, Picone [1] established the following identity.

Picone Identity

If $x(t)$ and $y(t)$ are the nontrivial solutions of

$$(p_1(t)x'(t))' + q_1(t)x(t) = 0,$$
$$(p_2(t)y'(t))' + q_2(t)y(t) = 0,$$

(1.3)

where $t \in [\alpha,\beta]$, $p_1'(t)$, $p_2'(t)$, $q_1(t)$, and $q_2(t)$ are real and continuous functions in $[\alpha,\beta]$. If $y(t) \neq 0$ for $t \in [\alpha,\beta]$, then

$$\left(\frac{x(t)}{y(t)}(p_1(t)x'(t)y(t) - p_2(t)y'(t)x(t))\right)'$$
$$= (p_1(t) - p_2(t))x'^2(t) + (q_2(t) - q_1(t))x^2(t) + p_2(t)\left(\frac{x(t)y'(t)}{y(t)} - x'(t)\right)^2.$$

(1.4)

By (1.4), one can easily obtain the Sturm comparison theorem of second-order linear differential equations (1.3).

Sturm-Picone Comparison Theorem

Assume that $x(t)$ and $y(t)$ are the nontrivial solutions of (1.3) and a, b are two consecutive zeros of $x(t)$, if

$$p_1(t) \geq p_2(t) > 0, \quad q_2(t) \geq q_1(t), \quad t \in [a,b],$$

(1.5)

then $y(t)$ has at least one zero on $[a,b]$.

Notes

Later, many mathematicians, such as Kamke, Leighton, and Reid [2–5] developed thier work. The investigation of Sturm comparison theorem has involved much interest in the new century [6, 7]. The Sturm comparison theorem of second-order difference equations

$$\begin{aligned} &\Delta[p_1(t-1)\Delta x(t-1)] + q_1(t)x(t) = 0, \\ &\Delta[p_2(t-1)\Delta y(t-1)] + q_2(t)y(t) = 0, \end{aligned} \tag{1.6}$$

has been investigated , where $p_1(t) \geq p_2(t) > 0$ on $[\alpha, \beta+1]$, $q_2(t) \geq q_1(t)$ on $[\alpha+1, \beta+1]$, α, β are integers, and Δ is the forward difference operator: $\Delta x(t) = x(t+1) - x(t)$. In 1995, Zhang [9] extended this result. But we will remark that in [8, Chapter 8] the authors employed the Riccati equation and a positive definite quadratic functional in their proof. Recently, the Sturm comparison theorem on time scales has received a lot of attentions. In [10, Chapter 4], the mathematicians studied

$$\begin{aligned} &(p_1(t)x^{\Delta}(t))^{\nabla} + q_1(t)x(t) = 0, \\ &(p_2(t)y^{\Delta}(t))^{\nabla} + q_2(t)y(t) = 0, \end{aligned} \tag{1.7}$$

where $p_1(t) \geq p_2(t) > 0$ and $q_2(t) \geq q_1(t)$ for $t \in [\rho(\alpha), \sigma(\beta)] \cap \mathbf{T}$, y^{∇} is the nabla derivative, and they get the Sturm comparison theorem. We will make use of Picone identity on time scales to prove the Sturm-Picone comparison theorem of (1.1) and (1.2).

This paper is organized as follows. Section 2 introduces some basic concepts and fundamental results about time scales, which will be used in Section 3. In Section 3 we first give the Picone identity on time scales, then we will employ this to prove our main result: Sturm-Picone comparison theorem of (1.1) and (1.2) on time scales.

13.2 PRELIMINARIES

In this section, some basic concepts and some fundamental results on time scales are introduced.

Let $\mathbf{T} \subset \mathbb{R}$ be a nonempty closed subset. Define the forward and backward jump operators $\sigma, \rho : \mathbf{T} \to \mathbf{T}$ by

$\sigma(t) = \inf\{s \in \mathbf{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbf{T} : s < t\},$

(2.1)

where $\inf \varnothing = \sup \mathbf{T}$, $\sup \varnothing = \inf \mathbf{T}$. A point $t \in \mathbf{T}$ is called right-scattered, right-dense, left-scattered, and left-dense if $\sigma(t) > t, \sigma(t) = t, \rho(t) < t$, and $\rho(t) = t$, respectively. We put $\mathbf{T}^k = \mathbf{T}$ if $\mathbf{T}$ is unbounded above and $\mathbf{T}^k = \mathbf{T} \setminus (\rho(\max \mathbf{T}), \max \mathbf{T}]$ otherwise. The graininess functions $\nu, \mu : \mathbf{T} \to [0, \infty)$ are defined by

$\mu(t) = \sigma(t) - t, \qquad \nu(t) = t - \rho(t)\,.$

(2.2)

Let f be a function defined on $\mathbf{T}$. f is said to be (delta) differentiable at $t \in \mathbf{T}^k$ provided there exists a constant a such that for any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbf{T}$ for some $\delta > 0$) with

$|f(\sigma(t)) - f(s) - a(\sigma(t) - s)| \le \varepsilon\, |\sigma(t) - s|\,, \quad \forall s \in U\,.$(2.3)

In this case, denote $f^{\Delta}(t) := a$. If f is (delta) differentiable for every $t \in \mathbf{T}^k$, then f is said to be (delta) differentiable on $\mathbf{T}$. If f is differentiable at $t \in \mathbf{T}^k$, then

$$f^{\Delta}(t) = \begin{cases} \lim\limits_{\substack{s \to t \\ s \in \mathbb{T}}} \dfrac{f(t) - f(s)}{t - s}, & \text{if } \mu(t) = 0, \\ \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}, & \text{if } \mu(t) > 0\,. \end{cases} \qquad (2.4)$$

If $F^{\Delta}(t) = f(t)$ for all $t \in \mathbf{T}^k$, then $F(t)$ is called an antiderivative of f on $\mathbf{T}$. In this case, define the delta integral by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s) \quad \forall s, t \in \mathbf{T}\,. \qquad (2.5)$$

Moreover, a function f defined on $\mathbf{T}$ is said to be rd-continuous if it is continuous at every right-dense point in $\mathbf{T}$ and its left-sided limit exists at every left-dense point in $\mathbf{T}$.

Lemma 2.1. Let $f, g : \mathbf{T} \to \mathbb{R}$ and $t \in \mathbf{T}^k$.

1. (i)If f is differentiable at t, then f is continuous at t.
2. (ii) If f and g are differentiable at t, then fg is differentiable at t and

Notes

$$(fg)^{\Delta}(t) = f^{\sigma}(t)g^{\Delta}(t) + f^{\Delta}(t)g(t) = f^{\Delta}(t)g^{\sigma}(t) + f(t)g^{\Delta}(t).$$
(2.6)

3. (iii) If f and g are differentiable at t, and $f(t)f^{\sigma}(t) \neq 0$, then $f^{-1}g$ is differentiable at t and

$$\left(gf^{-1}\right)^{\Delta}(t) = \left(g^{\Delta}(t)f(t) - g(t)f^{\Delta}(t)\right)\left(f^{\sigma}(t)f(t)\right)^{-1}. \qquad (2.7)$$

4. (iv) If f is rd-continuous on $\mathbf{T}$, then it has an antiderivative on $\mathbf{T}$.

Definition 2.2. A function $f:\mathbf{T} \to \mathbb{R}$ is said to be right-increasing at $t_0 \in \mathbf{T} \setminus \{\max \mathbf{T}\}$ provided

1. (i) $f(\sigma(t_0)) > f(t_0)$ in the case that t_0 is right-scattered;
2. (ii) there is a neighborhood U of t_0 such that $f(t) > f(t_0)$ for all $t \in U$ with $t > t_0$ in the case that t_0 is right-dense.

If the inequalities for f are reversed in (i) and (ii), f is said to be right-decreasing at t_0.

The following result can be directly derived from (2.4).

Lemma 2.3. Assume that $f:\mathbf{T} \to \mathbb{R}$ is differentiable at $t_0 \in \mathbf{T} \setminus \{\max \mathbf{T}\}$. If $f^{\Delta}(t_0) > 0$, then f is right-increasing at t_0; and if $f^{\Delta}(t_0) < 0$, then f is right-decreasing at t_0.

Definition 2.4. One says that a solution $x(t)$ of (1.1) has a generalized zero at t if $x(t) = 0$ or, if t is right-scattered and $x(t)x(\sigma(t)) < 0$. Especially, if $x(t)x(\sigma(t)) < 0$, then we say $x(t)$has a node at $(t + \sigma(t))/2$.

A function $p:\mathbf{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)p(t) \neq 0, \quad \forall t \in \mathbf{T}. \qquad (2.8)$$

Hilger [14] showed that for $t_0 \in \mathbf{T}$ and rd-continuous and regressive p, the solution of the initial value problem

$$y^{\Delta}(t) = p(t)y(t), \qquad y(t_0) = 1$$

(2.9)

is given by $e_p(\cdot, t_0)$, where

$$e_p(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \quad \text{with } \xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0 \\ z, & \text{if } h = 0\,. \end{cases}$$

(2.10)

The development of the theory uses similar arguments and the definition of the nabla derivative.

13.3 MAIN RESULTS

In this section, we give and prove the main results of this paper.

First, we will show that the following second-order linear equation:

$$x^{\Delta\Delta}(t) + a_1(t)x^{\Delta\sigma}(t) + a_2(t)x^{\sigma}(t) = 0$$

(3.1)

can be rewritten as (1.1).

Theorem 3.1. If $1+\mu(t)a_1(t) \neq 0$ and $a_2(t)$ is continuous, then (3.1) can be written in the form of (1.1), with

$$p_1(t) = e_{a_1}(t,t_0), \qquad q_1(t) = e_{a_1}(t,t_0)a_2(t)\,.$$

(3.2)

Proof. Multiplying both sides of (3.1) by $e_{a_1}(t,t_0)$, we get

$$\begin{aligned} 0 &= e_{a_1}(t,t_0)x^{\Delta\Delta}(t) + e_{a_1}(t,t_0)a_1(t)x^{\Delta\sigma}(t) + e_{a_1}(t,t_0)a_2(t)x^{\sigma}(t) \\ &= e_{a_1}(t,t_0)x^{\Delta\Delta}(t) + \left[e_{a_1}(t,t_0)\right]^{\Delta}x^{\Delta\sigma}(t) + e_{a_1}(t,t_0)a_2(t)x^{\sigma}(t) \\ &= \left[e_{a_1}(t,t_0)x^{\Delta}(t)\right]^{\Delta} + e_{a_1}(t,t_0)a_2(t)x^{\sigma}(t), \end{aligned}$$

(3.3)

where we used Lemma 2.1. This equation is in the form of (1.1) with $p_1(t)$ and $q_1(t)$ as desired.

Lemma 3.2 (Picone Identity). Let $x(t)$ and $y(t)$ be the nontrivial solutions of (1.1) and (1.2) with $p_1(t) \geq p_2(t) > 0$ and $q_2(t) \geq q_1(t)$ for $t \in [\alpha,\beta] \cap \mathbf{T}$. If $y(t)$ has no generalized zeros on $[\alpha,\beta] \cap \mathbf{T}$, then the following identity holds:

$$\left(\frac{x(t)}{y(t)}(p_1(t)x^\Delta(t)y(t) - p_2(t)y^\Delta(t)x(t))\right)^\Delta$$
$$= (p_1(t) - p_2(t))(x^\Delta(t))^2 + (q_2(t) - q_1(t))x^2(\sigma(t))$$
$$+\left(\sqrt{\frac{y(t)}{p_2(t)y(\sigma(t))}}\frac{p_2(t)y^\Delta(t)}{y(t)}x(\sigma(t)) - \sqrt{\frac{p_2(t)y(\sigma(t))}{y(t)}}x^\Delta(t)\right)^2.$$

(3.4)

Proof. We first divide the left part of (3.4) into two parts

$$\left(\frac{x(t)}{y(t)}(p_1(t)x^\Delta(t)y(t) - p_2(t)y^\Delta(t)x(t))\right)^\Delta = \left(p_1(t)x^\Delta(t)x(t) - \frac{p_2(t)y^\Delta(t)}{y(t)}x^2(t)\right)^\Delta$$
$$= (p_1(t)x^\Delta(t)x(t))^\Delta - \left(\frac{p_2(t)y^\Delta(t)}{y(t)}x^2(t)\right)^\Delta.$$

(3.5)

From (1.1) and the product rule (Lemma 2.1(ii), we have

$$(p_1(t)x^\Delta(t)x(t))^\Delta = (p_1(t)x^\Delta(t))^\Delta x(\sigma(t)) + p_1(t)x^\Delta(t)x^\Delta(t)$$
$$= p_1(t)(x^\Delta(t))^2 - q_1(t)x^2(\sigma(t)) \quad \forall\, t \in [\alpha,\beta] \cap \mathbf{T}.$$

(3.6)

It follows from (1.2), (2.4), product and quotient rules (Lemma 2.1(ii), (iii) and the assumption that $y(t)$ has no generalized zeros on $[\alpha,\beta] \cap \mathbf{T}$ that

$$\left(\frac{p_2(t)y^\Delta(t)}{y(t)}x^2(t)\right)^\Delta$$

$$= x^2(\sigma(t))\left(\frac{p_2(t)y^\Delta(t)}{y(t)}\right)^\Delta + x(\sigma(t))x^\Delta(t)\frac{p_2(t)y^\Delta(t)}{y(t)} + x^\Delta(t)x(t)\frac{p_2(t)y^\Delta(t)}{y(t)}$$

$$= x^2(\sigma(t))\left(-q_2(t) - p_2(t)\frac{(y^\Delta(t))^2}{y(t)y(\sigma(t))}\right) + x(\sigma(t))x^\Delta(t)\frac{p_2(t)y^\Delta(t)}{y(t)}$$

$$+x^\Delta(t)(x(\sigma(t)) - \mu(t)x^\Delta(t))\frac{p_2(t)y^\Delta(t)}{y(t)}$$

$$= p_2(t)(x^\Delta(t))^2 - q_2(t)x^2(\sigma(t)) - p_2(t)\frac{(y^\Delta(t))^2 x^2(\sigma(t))}{y(t)y(\sigma(t))}$$

$$+2x(\sigma(t))x^\Delta(t)\frac{p_2(t)y^\Delta(t)}{y(t)} - \left(p_2(t) + \mu(t)\frac{p_2(t)y^\Delta(t)}{y(t)}\right)(x^\Delta(t))^2$$

$$= p_2(t)(x^\Delta(t))^2 - q_2(t)x^2(\sigma(t)) - \frac{y(t)}{p_2(t)y(\sigma(t))}\left(\frac{p_2(t)y^\Delta(t)}{y(t)}\right)^2 x^2(\sigma(t))$$

$$+2x(\sigma(t))x^\Delta(t)\frac{p_2(t)y^\Delta(t)}{y(t)} - \frac{p_2(t)y(\sigma(t))}{y(t)}(x^\Delta(t))^2$$

$$= p_2(t)(x^\Delta(t))^2 - q_2(t)x^2(\sigma(t))$$

$$-\left(\sqrt{\frac{y(t)}{p_2(t)y(\sigma(t))}}\frac{p_2(t)y^\Delta(t)}{y(t)}x(\sigma(t)) - \sqrt{\frac{p_2(t)y(\sigma(t))}{y(t)}}x^\Delta(t)\right)^2 \quad \forall t \in [\alpha,\beta]\cap\mathbf{T}.$$

(3.7)

Combining $(p_1(t)x^\Delta(t)x(t))^\Delta$ and $-((p_2(t)y^\Delta(t)/y(t))x^2(t))^\Delta$, we get (3.4). This completes the proof.

Now, we turn to proving the main result of this paper.

Theorem 3.3 (Sturm-Picone Comparison Theorem).

Suppose that $x(t)$ and $y(t)$ are the nontrivial solutions of (1.1) and (1.2), and a, b are two consecutive generalized zeros of $x(t)$, if

$$p_1(t) \geq p_2(t) > 0, \quad q_2(t) \geq q_1(t), \quad t \in [a,b]\cap\mathbf{T},$$

(3.8)

then $y(t)$ has at least one generalized zero on $[a,b]\cap\mathbf{T}$.

Proof. Suppose to the contrary, $y(t)$ has no generalized zeros on $[a,b]\cap\mathbf{T}$ and $y(t) > 0$ for all $t \in [a,b]\cap\mathbf{T}$.

Case 1. Suppose a, b are two consecutive zeros of $x(t)$. Then by Lemma 3.2, (3.4) holds and integrating it from a to b we get

$$\int_a^b \left(\frac{x(t)}{y(t)}(p_1(t)x^\Delta(t)y(t)-p_2(t)y^\Delta(t)x(t))\right)^\Delta \Delta t$$
$$=\int_a^b \Bigg((p_1(t)-p_2(t))(x^\Delta(t))^2+(q_2(t)-q_1(t))x^2(\sigma(t))$$
$$+\left(\sqrt{\frac{y(t)}{p_2(t)y(\sigma(t))}}\frac{p_2(t)y^\Delta(t)}{y(t)}-\sqrt{\frac{p_2(t)y(\sigma(t))}{y(t)}}x^\Delta(t)\right)^2\Bigg)\Delta t.$$

(3.9)

Noting that $x(a)=x(b)=0$, we have

$$\int_a^b \left(\frac{x(t)}{y(t)}(p_1(t)x^\Delta(t)y(t)-p_2(t)y^\Delta(t)x(t))\right)^\Delta \Delta t$$
$$=\left(\frac{x(t)}{y(t)}(p_1(t)x^\Delta(t)y(t)-p_2(t)y^\Delta(t)x(t))\right)\Bigg|_a^b$$
$$=0.$$

(3.10)

Hence, by (3.9) and $p_1(t)\ge p_2(t)>0,\ q_2(t)\ge q_1(t)$, for all $t\in[a,b]\cap\mathbf{T}$ we have

$$0\ =\int_a^b \Bigg((p_1(t)-p_2(t))(x^\Delta(t))^2+(q_2(t)-q_1(t))x^2(\sigma(t))$$
$$+\left(\sqrt{\frac{y(t)}{p_2(t)y(\sigma(t))}}\frac{p_2(t)y^\Delta(t)}{y(t)}-\sqrt{\frac{p_2(t)y(\sigma(t))}{y(t)}}x^\Delta(t)\right)^2\Bigg)\Delta t$$
$$>0,$$

(3.11)

which is a contradiction. Therefore, in Case 1, $y(t)$ has at least one generalized zero on $[a,b]\cap\mathbf{T}$.

Case 2. Suppose a is a zero of $x(t)$, $(b+\sigma(b))/2$ is a node of $x(t)$, $x(b)<0$, and $x(\sigma(b))>0$. It follows from the assumption that $y(t)$ has no generalized zeros on $[a,b]\cap\mathbf{T}$ and that $y(t)>0$ for all $t\in[a,b]\cap\mathbf{T}$ that $y(\sigma(b))>0$. Hence by (2.4) and $p_2(t)\ge p_1(t)>0$ on $[a,b]\cap\mathbf{T}$, we have

$$\frac{x(b)}{y(b)}(p_1(b)x^\Delta(b)y(b)-p_2(b)y^\Delta(b)x(b))$$
$$=\frac{x(b)}{y(b)}\frac{1}{\mu(b)}(p_1(b)x(\sigma(b))y(b)-p_2(b)y(\sigma(b))x(b)+(p_2(b)-p_1(b))x(b)y(b))$$
$$<0.$$

(3.12)

By integration, it follows from (3.12) and $x(a)=0$ that

$$\int_a^b \left(\frac{x(t)}{y(t)}(p_1(t)x^{\Delta}(t)y(t)-p_2(t)y^{\Delta}(t)x(t))\right)^{\Delta}\Delta t$$
$$=\left(\frac{x(t)}{y(t)}(p_1(t)x^{\Delta}(t)y(t)-p_2(t)y^{\Delta}(t)x(t))\right)\Bigg|_a^b$$
$$=\frac{x(b)}{y(b)}(p_1(b)x^{\Delta}(b)y(b)-p_2(b)y^{\Delta}(b)x(b))$$
$$<0.$$

(3.13)

So, from (3.9) and above argument we obtain that

$$0>\int_a^b\Bigg((p_1(t)-p_2(t))(x^{\Delta}(t))^2+(q_2(t)-q_1(t))x^2(\sigma(t))$$
$$+\left(\sqrt{\frac{y(t)}{p_2(t)y(\sigma(t))}}\frac{p_2(t)y^{\Delta}(t)}{y(t)}-\sqrt{\frac{p_2(t)y(\sigma(t))}{y(t)}}x^{\Delta}(t)\right)^2\Bigg)\Delta t$$
$$>0,$$

(3.14)

which is a contradiction, too. Hence, in Case 2, $y(t)$ has at least one generalized zero on $[a,b]\cap\mathbf{T}$.

Case 3. Suppose $(a+\sigma(a))/2$ is a node of $x(t), x(a)>0, x(\sigma(a))<0,$ and b is a generalized zero of $x(t)$. Similar to the discussion of (3.12), we have

$$\frac{x(a)}{y(a)}(p_1(a)x^{\Delta}(a)y(a)-p_2(a)y^{\Delta}(a)x(a))$$
$$=\frac{x(a)}{y(a)}\frac{1}{\mu(a)}(p_1(a)x(\sigma(a))y(a)-p_2(a)y(\sigma(a))x(a)+(p_2(a)-p_1(a))x(a)y(a))$$
$$<0,$$

(3.15)

which implies

$$(p_1(a)x^{\Delta}(a)y(a)-p_2(a)y^{\Delta}(a)x(a))<0.$$

(3.16)

(i)If $(b+\sigma(b))/2$ is a node of $x(t),$ then $x(b)<0, x(\sigma(b))>0$. Hence, we have (3.12), that is,

$$\frac{x(b)}{y(b)}(p_1(b)x^{\Delta}(b)y(b)-p_2(b)y^{\Delta}(b)x(b))<0.$$

(3.17)

Notes

(ii)If b is a zero of $x(t)$, then

$$\frac{x(b)}{y(b)}(p_1(b)x^{\Delta}(b)y(b)-p_2(b)y^{\Delta}(b)x(b))=0\,.$$

(3.18)

It follows from (3.4) and Lemma 2.3 that

$$\frac{x(t)}{y(t)}(p_1(t)x^{\Delta}(t)y(t)-p_2(t)y^{\Delta}(t)x(t))$$

(3.19)

is right-increasing on $[a,b]\cap\mathbf{T}$. Hence, from (i) and (ii) that

$$\begin{aligned}&\frac{x(a)}{y(a)}(p_1(a)x^{\Delta}(a)y(a)-p_2(a)y^{\Delta}(a)x(a))\\&\quad<\frac{x(\sigma(a))}{y(\sigma(a))}(p_1(\sigma(a))x^{\Delta}(\sigma(a))y(\sigma(a))-p_2(\sigma(a))y^{\Delta}(\sigma(a))x(\sigma(a)))\\&\quad<0,\end{aligned}$$

(3.20)

which implies

$$p_1(\sigma(a))x^{\Delta}(\sigma(a))y(\sigma(a))-p_2(\sigma(a))y^{\Delta}(\sigma(a))x(\sigma(a))>0\,.$$

(3.21)

From (3.16), (3.21), and (2.4), we have

$$(p_1x^{\Delta}y-p_2y^{\Delta}x)^{\Delta}(a)=\frac{1}{\mu(a)}((p_1x^{\Delta}y-p_2y^{\Delta}x)(\sigma(a))-(p_1x^{\Delta}y-p_2y^{\Delta}x)(a))>0\,.$$

(3.22)

Further, it follows from (1.1), (1.2), product rule (Lemma 2.1(ii), and (3.22) that

$$(p_1x^{\Delta}y-p_2y^{\Delta}x)^{\Delta}(a)=(q_2(a)-q_1(a))x(\sigma(a))y(\sigma(a))+(p_1(a)-p_2(a))x^{\Delta}(a)y^{\Delta}(a)>0\,.$$

(3.23)

If $p_1(a)=p_2(a)$ and from $q_2(a)\geq q_1(a)$, $x(\sigma(a))<0$, and $y(\sigma(a))>0$ we have

$$(q_2(a)-q_1(a))x(\sigma(a))y(\sigma(a))<0\,.$$

(3.24)

This contradicts (3.22). Note that $x^{\Delta}(a)=(1/\mu(a))(x(\sigma(a))-x(a))$. It follows from $p_1(a)>p_2(a)>0$, (3.23), and (3.24) that

$y^{\Delta}(a) < 0$.

(3.25)

On the other hand, it follows from $x(t)$ and $y(t)$ are solutions of (1.1) and (1.2) that

$$y(\sigma(a))\left((p_1(a)x^{\Delta}(a))^{\Delta} + q_1(a)x(\sigma(a))\right) = 0,$$
$$x(\sigma(a))\left((p_2(a)y^{\Delta}(a))^{\Delta} + q_2(a)y(\sigma(a))\right) = 0.$$

(3.26)

Combining the above two equations we obtain

$$\left((p_1(a)x^{\Delta}(a))^{\Delta}y(\sigma(a)) - (p_2(a)y^{\Delta}(a))^{\Delta}x(\sigma(a))\right) + (q_1(a) - q_2(a))x(\sigma(a))y(\sigma(a)) = 0.$$

(3.27)

It follows from (3.27) and (2.4) that

$$\begin{aligned}
&\frac{1}{\mu(a)}\{[p_1(\sigma(a))x^{\Delta}(\sigma(a)) - p_1(a)x^{\Delta}(a)]y(\sigma(a)) - [p_2(\sigma(a))y^{\Delta}(\sigma(a)) - p_2(a)y^{\Delta}(a)]x(\sigma(a))\} \\
&\qquad +(q_1(a) - q_2(a))x(\sigma(a))y(\sigma(a)) \\
&\quad = \frac{1}{\mu(a)}[p_2(a)y^{\Delta}(a)x(\sigma(a)) - p_1(a)x^{\Delta}(a)y(\sigma(a))] \\
&\qquad +\frac{1}{\mu(a)}[p_1(\sigma(a))x^{\Delta}(\sigma(a))y(\sigma(a)) - p_2(\sigma(a))y^{\Delta}(\sigma(a))x(\sigma(a))] \\
&\qquad +(q_1(a) - q_2(a))x(\sigma(a))y(\sigma(a)) \\
&\quad = 0.
\end{aligned}$$

(3.28)

Hence, from $q_2(a) \geq q_1(a)$, $x(\sigma(a)) < 0$, $y(\sigma(a)) > 0$, and (3.21), we get

$$p_2(a)y^{\Delta}(a)x(\sigma(a)) - p_1(a)x^{\Delta}(a)y(\sigma(a)) < 0.$$

(3.29)

By referring to $x^{\Delta}(a) < 0$ and $p_1(a) > p_2(a) > 0$, it follows that

$$y^{\Delta}(a) > 0,$$

(3.30)

which contradicts $y^{\Delta}(a) < 0$.

It follows from the above discussion that $y(t)$ has at least one generalized zero on $[a,b] \cap \mathbf{T}$. This completes the proof.

Notes

Remark 3.4. If $p_1(t) \equiv p_2(t) \equiv 1,$ then Theorem 3.3 reduces to classical Sturm comparison theorem.

Remark 3.5. In the continuous case: $\mu(t) \equiv 0$. This result is the same as Sturm-Picone comparison theorem of second-order differential equations (see Section 1).

Remark 3.6. In the discrete case: $\mu(t) \equiv 1$. This result is the same as Sturm comparison theorem of second-order difference equations .

Example 3.7. Consider the following three specific cases:

$$[0,1] \cap \mathbf{T} = \left[0, \frac{1}{2}\right] \cup \left[\frac{2}{3}, 1\right],$$
$$[0,1] \cap \mathbf{T} = \left[0, \frac{1}{2}\right] \cup \left\{\frac{1}{2(N-1)}, \frac{1}{(N-1)}, \frac{3}{2(N-1)}, \ldots, 1\right\}, \quad N > 2,$$
$$[0,1] \cap \mathbf{T} = \left\{q^k \mid k \geq 0, k \in \mathbb{Z}\right\} \cup \{0\}, \quad \text{where } 0 < q < 1 .$$

(3.31)

By Theorem 3.3, we have if $x(t)$ and $y(t)$ are the nontrivial solutions of (1.1) and (1.2), a, b are two consecutive generalized zeros of $x(t)$, and $p_1(t) \geq p_2(t) > 0,\ q_2(t) \geq q_1(t),\ t \in [a,b] \cap \mathbf{T},$ then $y(t)$ has at least one generalized zero on $[a,b] \cap \mathbf{T}$. Obviously, the above three cases are not continuous and not discrete. So the existing results for the differential and difference equations are not available now.

By Remarks 3.4–3.6 and Example 3.7, the Sturm comparison theorem on time scales not only unifies the results in both the continuous and the discrete cases but also contains more complicated time scales.

Check In Progress-I

Q. 1 Write Picano Identity Lemma .

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

Q.2 State **Sturm-Picone Comparison Theorem**.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

13.4 STURM-LIOUVILLE EQUATION

A second-order ordinary differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[\lambda w(x)-q(x)]\,y=0,$$

where λ is a constant and $w(x)$ is a known function called either the density or weighting function. The solutions (with appropriate boundary conditions) of λ are called eigenvalues and the corresponding $u_\lambda(x)$ eigenfunctions. The solutions of this equation satisfy important mathematical properties under appropriate boundary conditions (Arfken 1985).

There are many approaches to solving Sturm-Liouville problems in the Wolfram Language. Probably the most straightforward approach is to use variational (or Galerkin) methods. For example, VariationalBound in the Wolfram
Language package VariationalMethods` and NVariationalBound give approximate eigenvalues and eigenfunctions.

A problem generated by the following equation, where x varies in a given finite or infinite interval (a, b),

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+l(x)y=\lambda r(x)y,$$

together with some boundary conditions, where $p(x)$ and $r(x)$ are positive, $l(x)$ is real and λ is a complex parameter. Serious studies of this problem were started by J.Ch. Sturm and J. Liouville. The methods and notions that originated during studies of the Sturm–Liouville problem played an important role in the development of many directions in mathematics and physics. It was and remains a constant source of new ideas and problems in the spectral theory of operators and in related problems in analysis. Recently it gained even greater significance, when its relation to certain non-linear evolution equations of mathematical physics were discovered.

If $p(x)$ is differentiable and $p(x)r(x)$ is twice differentiable, then, by a substitution, equation (1) can be reduced to (see [1])

$$-y''+q(x)y=\lambda y.$$

It is customary to distinguish between regular and singular problems. A Sturm–Liouville problem for equation (2) is called regular if the interval (a, b) in which x varies is finite and if the function $q(x)$ is summable on the entire interval (a, b). If the interval (a, b) is infinite or if $q(x)$ is not summable (or both), then the problem is called singular.

Below the following possibilities will be considered in some detail: 1) the interval (a, b) is finite (in this case, without loss of generality, one may assume $a=0$ and $b=\pi$); 2) $a=0$, $b=\infty$; or 3) $a=-\infty$, $b=\infty$.

1. Consider the problem given on the interval $[0, \pi]$ by equation (2) and the separated boundary conditions

$$y'(0)-hy(0)=0, \qquad y'(\pi)+Hy(\pi)=0,$$

where $q(x)$ is a real summable function on $[0, \pi]$, h and H are arbitrary finite or infinite fixed real numbers and λ is a complex parameter. If $h=\infty$ $(H=\infty)$, then the first (second) condition in (3) is replaced by $y(0)=0$ $(y(\pi)=0)$. To be specific it is further assumed that all numbers occurring in the boundary conditions are finite.

The number λ_0 is called an eigen value for the problem (2), (3) if for $\lambda = \lambda_0$ equation (2) has a non-trivial solution $y_0(x) \not\equiv 0$ that satisfies (3); the function $y_0(x)$ is then called the eigenfunction corresponding to the eigenvalue λ_0.

The eigenvalues for the boundary value problem (2), (3) are real; to the distinct eigenvalues correspond linearly independent eigenfunctions (since $q(x)$ and the numbers h, H are real, the eigenfunctions for the problem (2), (3) can be chosen to be real); eigenfunctions $y_1(x)$ and $y_2(x)$ corresponding to different eigenvalues are unique and orthogonal, i.e. $\int_0^\pi y_1(x) y_2(x)\,dx = 0$.

There exists an unboundedly-increasing sequence of eigenvalues $\lambda_0, \lambda_1, \dots$ for the boundary value problem (2), (3); moreover, the eigenfunction $y_n(x)$ corresponding to the eigenvalue λ_n has precisely n zeros in the interval $(0, \pi)$.

Let $W_2^m[0, \pi]$ be the Sobolev space of complex-valued functions on the interval $[0, \pi]$ that have $m-1$ absolutely-continuous derivatives and with m-th derivatives summable on $[0, \pi]$. If $q \in W_2^m[0, \pi]$, then the eigenvalues λ_n of the boundary value problem (2), (3) for large n satisfy the following asymptotic equation (see [4]):

$$\sqrt{\lambda_n} = n + \sum_{1 \le 2j+1 \le m+2} \frac{c_{2j+1}}{n^{2j+1}} +$$
$$+ \frac{(-1)^{m-1}}{2^{m+2}} \left(S_m(n) + \frac{\tilde{S}_m(n)}{n} \right) \frac{1}{n^{m+1}} + \frac{\delta_n}{n^{m+2}} + \frac{\epsilon_n(h, H)}{n^{m+2}},$$

where c_{2j+1} are numbers independent of n,

$$c_1 = \frac{1}{\pi}\left(h + H + \frac{1}{2}\int_0^\pi q(t)\,dt \right),$$

$$S_m(n) = \frac{2}{\pi}\int_0^\pi q^{(m)}(t) \sin\left\{ 2nt - \frac{\pi}{2}(m+1) \right\} dt,$$

$$\tilde{S}_m(n) = \frac{2}{\pi}\int_0^\pi q^{(m)}(t)(2h - c_1 t) \sin\left\{ 2nt - \frac{\pi}{2}(m+2) \right\} dt,$$

δ_n does not depend on h, H, and

$$\sum_{n=0}^{\infty} |\delta_n|^2 < \infty, \qquad \sum_{n=0}^{\infty} |\epsilon_n(h,H)|^2 < \infty .$$

The above implies, in particular, that if $q \in W_2^1[0, \pi]$, then

$$\lambda_n = n^2 + c + \frac{\gamma_n}{n},$$

where

$$c = \frac{2}{\pi}\left(h + H + \frac{1}{2}\int_0^{\pi} q(t)\,dt\right), \qquad \sum_{n=0}^{\infty} |\gamma_n|^2 < \infty .$$

Thus, the series $\sum_{n=0}^{\infty}(\lambda_n - n^2 - c)$ is convergent. Its sum is called the regularized trace of the problem (2), (3) (see [13]):

$$\sum_{n=0}^{\infty}(\lambda - n^2 - c) = \frac{q(0)+q(\pi)}{4} - \frac{(h+H)^2}{2} + hH - \frac{c}{2}.$$

Let $v_0(x), v_1(x), \ldots,$ be the orthonormal eigenfunctions of the problem (2), (3) corresponding to the eigenvalues $\lambda_0, \lambda_1, \ldots$. For any function $f \in L_2[0, \pi]$ the so-called Parseval equality holds:

$$\int_0^{\pi} |f(x)|^2\,dx = \sum_{n=0}^{\infty} |a_n|^2,$$

where

$$a_n = \int_0^{\pi} f(x) v_n(x)\,dx,$$

and the following formula for eigen-function expansion is valid:

$$f(x) = \sum_{n=0}^{\infty} a_n v_n(x),$$

where the series converges in the metric of $L_2[0, \pi]$. Completeness and expansion theorems for a regular Sturm–Liouville problem were first proved by V.A. Steklov [14].

If the function f has a continuous second derivative and satisfies the boundary conditions (3), then the following assertions hold (see [15]):

a) the series (4) converges absolutely and uniformly on $[0, \pi]$ to $f(x)$;

b) the once-differentiated series (4) converges absolutely and uniformly on $[0, \pi]$ to $f'(x)$;

c) at any point where $f''(x)$ satisfies some local condition of expansion in a Fourier series (e.g. is of bounded variation), the twice-differentiated series (4) converges to $f''(x)$.

For any function $f \in L_1[0, \pi]$ the series (4) is uniformly equiconvergent with the Fourier cosine series of f, i.e.

$$\lim_{N\to\infty} \sup_{0\le x\le\pi} |V_{N,f}(x) - c_{N,f}(x)| = 0,$$

where

$$V_{N,f}(x) = \int_0^\pi f(t) \sum_{n=0}^{N} v_n(x) v_n(t)\, dt,$$

$$c_{N,f}(x) = \int_0^\pi f(t) \left\{ \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{N} \cos nx \cos nt \right\} dt.$$

This means that the expansion of f with respect to the eigenfunctions of the boundary value problem (2), (3) converges under the same conditions as the expansion of f in a Fourier cosine series.

2 . The differential equation (2) is considered on the half-line $0 \le x < \infty$ with a boundary condition at zero:

$$y'(0) - hy(0) = 0 .$$

The function q is assumed to be real and summable on any finite subinterval of $[0, \infty)$ and h is assumed to be real.

Let $\phi(x, \lambda)$ be a solution of (2) with the initial conditions $y(0) = 1$, $y'(0) = h$ (so that $\phi(x, \lambda)$ satisfies also the boundary condition (5)). Let f be any function from $L_2(0, \infty)$ and let $\Phi_{f,b}(x) = \int_0^b f(x) \phi(x, \lambda)\, dx$, where b is an arbitrary finite positive number. For any function q and any number h there is at least one decreasing function $\rho(\lambda)$, $-\infty < \lambda < \infty$, independent of f, that has the following properties:

a) there is a function $\Phi_f(\lambda)$, which is the limit of $\Phi_{f,b}(\lambda)$ for $b \to \infty$ in the metric of $L_{2,\rho}(-\infty, \infty)$ (the space of ρ -measurable functions $\psi(\lambda)$ for which $\|\psi\|^2 = \int_{-\infty}^{\infty} |\psi(\lambda)|^2 d\rho(\lambda) < \infty$), i.e.

$$\lim_{b \to \infty} \int_{-\infty}^{\infty} |\Phi_f(\lambda) - \Phi_{f,b}(\lambda)|^2 d\rho(\lambda) = 0;$$

b) the Parseval equality is valid:

$$\int_0^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\Phi_f(\lambda)|^2 d\rho(\lambda).$$

The function $\rho(\lambda)$ is called the spectral function (or spectral density) for the boundary value problem (2), (5) (see [9]–[11]).

For the spectral function $\rho(\lambda)$ of the problem (2), (5) the following asymptotic formula is true (for a more precise form, see):

$$\lim_{\lambda \to -\infty} e^{\sqrt{\lambda} x}(\rho(\lambda) - \rho(-\infty)) = 0, \quad 0 \le x < \infty,$$
$$\lim_{\lambda \to \infty} \left(\rho(\lambda) - \rho(-\infty) - \frac{2}{\pi}\sqrt{\lambda} + h\right) = 0.$$

The following equiconvergence theorem is valid : For an arbitrary function $f \in L_2(0, \infty)$, let

$$\Phi_f(\lambda) = \int_0^{\infty} f(x)\phi(x, \lambda)\, dx,$$

$$C_f(\lambda) = \int_0^{\infty} f(x) \cos\sqrt{\lambda}\, x\, dx$$

(the integrals converge in the metrics of $L_{2,\rho}(-\infty, \infty)$ and $L_{2,\sqrt{\lambda}}(0, \infty)$, respectively); then for any fixed $b < \infty$ the integral

$$\int_{-\infty}^{b} \Phi_f(\lambda)\phi(x, \lambda)\, d\rho(\lambda)$$

converges absolutely and uniformly with respect to $x \in [0, b]$, and

$$\lim_{N\to\infty}\sup_{0\le x<b}\left|\int_{-\infty}^{N}\Phi_f(\lambda)\phi(x,\lambda)\,d\rho(\lambda)+\right.$$

$$\left.-\frac{2}{\pi}\int_0^N C_f(\lambda)\cos\sqrt{\lambda}x\,d\sqrt{\lambda}\right|=0\,.$$

Let problem (2), (5) have a discrete spectrum, i.e. let its spectrum consist of a countable number of eigenvalues $\lambda_1 < \lambda_2 < \dots$ with a unique limit point at infinity. Under certain restrictions on the function q, for the function $N(\lambda)=\sum_{\lambda_n<\lambda} 1$, i.e. the number of eigenvalues less than λ, the following asymptotic formula is valid:

$$N(\lambda)\sim\frac{1}{2\pi}\int_{q(x)<\lambda}(\lambda-q(x))^{1/2}\,dx\,.$$

Simultaneously with $\phi(x,\lambda)$, a second solution $\theta(x,\lambda)$ of equation (2) is introduced, satisfying the conditions $\theta(0,\lambda)=0$, $\theta'(0,\lambda)=1$, so that $\phi(x,\lambda)$ and $\theta(x,\lambda)$ form a fundamental system of solutions of (2). For a fixed λ ($\operatorname{Im}\lambda\neq 0$) and $b>0$ the following fractional-linear function is considered:

$$w_{\lambda,b}=w_{\lambda,b}(t)=\frac{-\theta'(b,\lambda)-t\theta(b,\lambda)}{\phi'(b,\lambda)+t\phi(b,\lambda)}\,.$$

When the independent variable t varies on the real line, the point $w_{\lambda,b}$ describes a circle bounding a disc $K_{\lambda,b}$. It always lies in the same half-plane (lower or upper) as λ. When b increases, $K_{\lambda,b}$ shrinks, i.e. for $b<b'$ the disc $K_{\lambda,b'}$ lies entirely inside the disc $K_{\lambda,b}$. There is (for $b\to\infty$) a limit disc or a point $K_{\lambda,\infty}$; if

$$\int_0^\infty|\phi(x,\lambda)|^2\,dx<\infty,$$

then $K_{\lambda,\infty}$ is a disc, otherwise it is a point. If condition (6) is fulfilled for some non-real value of λ, then it is fulfilled for all values of λ. In the case of a limit disc, for any value of λ all solutions of (2) belong to $L_2(0,\infty)$, and in the case of a limit point, for any non-real value

Notes

of λ this equation has the solution $\theta(x, \lambda) + m(\lambda)\phi(x, \lambda)$, which belongs to $L_2(0, \infty)$, where $m(\lambda)$ is the limit point $(m(\lambda) = K_{\lambda, \infty})$.

If $q(x) \geq -cx^2$, where c is some positive constant, then the case of a limit point holds.

3. Consider now equation (2) on the whole line $-\infty < x < \infty$ under the assumption that $q(x)$ is a real summable function on every finite subinterval of $(-\infty, \infty)$. Let $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ be the solutions of (2) satisfying the conditions $\phi_1(0, \lambda) = \phi_2'(0, \lambda) = 1$, $\phi_1'(0, \lambda) = \phi_2(0, \lambda) = 0$.

There is at least one real symmetric non-decreasing matrix-function

$$\mathcal{P}(\lambda) = \begin{Vmatrix} \rho_{11}(\lambda) & \rho_{12}(\lambda) \\ \rho_{21}(\lambda) & \rho_{22}(\lambda) \end{Vmatrix}, \qquad -\infty < \lambda < \infty,$$

with the following properties:

a) for any function $f \in L_2(-\infty, \infty)$ there exist functions $\Phi_{j,f}(\lambda)$, $j = 1, 2$, defined by

$$\Phi_{j,f}(\lambda) = \lim_{b \to \infty} \int_{-b}^{b} f(x)\phi_j(x, \lambda)\,dx, \qquad j = 1, 2,$$

where the limit is in the metric of $L_{2,\mathcal{P}}(-\infty, \infty)$;

b) the Parseval equality is valid:

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \sum_{j,k=1}^{2} \int_{-\infty}^{\infty} \Phi_{j,f}(\lambda)\overline{\Phi_k(\lambda)}\,d\rho_{jk}(\lambda).$$

13.5 DISCONJUGACY

An nth order homogeneous linear differential operator (equation)

$$Ly \equiv y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$

is called disconjugate on an interval I if no non-trivial solution has n zeros on I, multiple zeros being counted according to their multiplicity. (In the Russian literature this is called non-oscillation on I; cf. also Oscillating solution; Oscillating differential equation.) If (a1) has a solution with n zeros on an interval, then the infimum of all values c, $c > a$, such that some solution has n zeros on $[a, c]$ is called the conjugate point of a and is denoted by $\eta(a)$. This infimum is achieved by a solution which has a total of at least n zeros at a and $\eta(a)$ and is positive on $(a, \eta(a))$. If the equation has continuous coefficients, the conjugate point $\eta(a)$ is a strictly increasing, continuous function of a. The adjoint equation has the same conjugate point as (a1). For general properties, see [a1], [a7].

There are numerous explicit sufficient criteria for the equation (a1) to be disconjugate. Many of them are of the form

$$\sum_{k=1}^{n} c_k (b-a)^k \|p_k\| < 1,$$

where $\|p_k\|$ is some norm of p_k, $I = [a, b]$ and c_k are suitable constants. These are "smallness conditions" which express the proximity of (a1) to the disconjugate equation $y^{(n)} = 0$. See [a12].

L is disconjugate on $[a, b]$ if and only if it has there the Pólya factorization

$$Ly \equiv \rho_n \frac{d}{dx}\left(\rho_{n-1} \cdots \frac{d}{dx}\left(\rho_1 \frac{d}{dx}(\rho_0\, y)\right)\cdots\right),\ \rho_i > 0,$$

or the equivalent Mammana factorization

$$Ly = \left(\frac{d}{dx} + r_n\right)\cdots\left(\frac{d}{dx} + r_1\right)y.$$

Among the various Pólya factorizations, the most important is the Trench canonical form [a11]: If L is disconjugate on (a, b), $b \leq \infty$, then there is essentially one factorization such that $\int^b \rho_i^{-1} = \infty$, $i = 1, \dots, n-1$.

Disconjugacy is closely related to solvability of the de la Vallée-Poussin multiple-point problem $Ly = g$, $y^{(i)}(x_j) = a_{ij}$, $i = 0, \dots, r_j - 1$

Notes

, $\sum_1^m r_j = n$. The Green's function of a disconjugate operator L and the related homogeneous boundary value problem satisfies

$$\frac{G(x,t)}{(x-x_1)^{r_1}\ldots(x-x_m)^{r_m}} > 0$$

for $x_1 \leq x \leq x_m$, $x_1 < t < x_m$ [a7]. Another interesting boundary value problem is the focal boundary value problem $y^{(i)}(x_j) = 0$, $i = r_{j-1}$,..., $r_j - 1$, $j = 1$,..., m, $0 = r_0 < r_1 < \ldots < r_m = n-1$.

For a second-order equation, the Sturm separation theorem (cf. Sturm theorem) yields that non-oscillation (i.e., no solution has a sequence of zeros converging to $+\infty$) implies that there exists a point a such that (a1) is disconjugate on $[a, \infty)$. For equations of order $n > 2$ this conclusion holds for a class of equations [a2] but not for all equations [a4].

Particular results about disconjugacy exist for various special types of differential equations.

1) The Sturm–Liouville operator (cf. Sturm–Liouville equation)

$$(py')' + qy = 0, \; p > 0,$$

has been studied using the Sturm (and Sturm–Picone) comparison theorem, the Prüfer transformation and the Riccati equation $z' + q + z^2 / p = 0$. It is also closely related to the positive definiteness of the quadratic functional $\int_a^b (py'^2 - qy^2)$. See [a10], [a1], [a5]. For example, (a2) is disconjugate on $[a, b]$ if $\int_a^b p^{-1} \times \int_a^b |q| < 4$.

2) Third-order equations are studied in [a3].

3) For a self-adjoint differential equation $\sum_{i=0}^m (p_{m-i} y^{(i)})^{(i)} = 0$, the existence of a solution with two zeros of multiplicity m has been studied. Their absence is called (m, m)-disconjugacy.

4) Disconjugacy of the analytic equation $w' + p(z)w = 0$ in a complex domain is connected to the theory of univalent functions. See [a8], [a5] and Univalent function.

5) Many particularly elegant result are available for two-term equations $y^{(n)} + p(x)y = 0$ and their generalizations $Ly + p(x)y = 0$ [a6], [a2].

Disconjugacy has also been studied for certain second-order linear differential systems of higher dimension [a1], [a9]. In the historical prologue of [a9], the connection to the calculus of variations (cf. also Variational calculus) is explained. The concepts of disconjugacy and oscillation have also been generalized to non-linear differential equations and functional-differential equations.

Check In Progress-II

Q. 1 State Sturm-Liouville Equation.

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

Q.2 Define Sturm–Liouville operator in brief .

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

13.6 OSCILLATING DIFFERENTIAL EQUATION

An ordinary differential equation which has at least one oscillating solution. There are different concepts of the oscillation of a solution. The most widespread are oscillation at a point (usually taken to be $+\infty$) and oscillation on an interval. A non-zero solution of the equation

$$u^{(n)}=f(t,u,u',\dots,u^{(n-1)}), \qquad n\geq 2,$$

where $f(t,0,\dots,0)=0$, is called oscillating at the point $+\infty$ (or on an interval I) if it has a sequence of zeros which converges to $+\infty$ (respectively, there are at least n zeros in I counted according to their multiplicity). Equation (1) is oscillating at $+\infty$ or on an interval I if its solutions are oscillating (at $+\infty$, respectively, on I).

Among equations which are oscillatory at $+\infty$ the equations which possess the properties A or B, i.e. which are compatible in a specific sense with one of the equations

$$u^{(n)}=-u \quad \text{or} \quad u^{(n)}=u,$$

are distinguished. Equation (1) is said to possess property A if all its solutions defined in a neighbourhood of $+\infty$ are oscillating when n is even; when n is odd, they should either be oscillating or satisfy the condition

$$\lim_{t\to+\infty} u^{(i-1)}(t)=0, \qquad i=1,\dots,n.$$

If every solution of equation (1) defined in a neighbourhood of $+\infty$, when n is even, is either oscillating, or satisfies condition (2) or

$$\lim_{t\to+\infty} |u^{(i-1)}(t)|=+\infty, \qquad i=1,\dots,n,$$

while when n is odd, it is either oscillating or satisfies condition (3), then the equation possesses property B.

The linear equation

$$u^{(n)}=a(t)u$$

with a locally summable coefficient $a\colon [t_0,+\infty)\to \mathbf{R}$ possesses property A (property B) if

$$a(t)\leq 0 \quad (a(t)\geq 0) \quad \text{when } t\geq t_0$$

and either

$$\int_{t_0}^{+\infty} t^{n-1-\epsilon}|a(t)|\,dt = +\infty$$

or

$$a(t) \leq \frac{\mu_n - \epsilon}{t^n} \qquad \left(a(t) \geq \frac{\nu_n + \epsilon}{t^n}\right)$$

when $t \geq t_0$, where $\epsilon > 0$ and μ_n is the smallest (ν_n is the largest) of the local minima (maxima) of the polynomial $x(x-1)\ldots(x-n+1)$.

An equation of Emden–Fowler type

$$u^{(n)} = a(t)|u|^{\lambda} \operatorname{sign} u, \qquad \lambda > 0, \qquad \lambda \neq 1,$$

with a locally summable non-positive (non-negative) coefficient $a\colon [t_0, +\infty) \to \mathbf{R}$ possesses property A (property B) if and only if

$$\int_{t_0}^{+\infty} t^{\mu}|a(t)|\,dt = +\infty,$$

where $\mu = \min\{n-1, (n-1)\lambda\}$.

In a number of cases the question of the oscillation of equation (1) can be reduced to the same question for the standard equations of the form (4) and (5) using a comparison theorem.

In studying the oscillatory properties of equations with deviating argument, certain specific features arise. For example, if n is odd, $\Delta > 0$, and if for large t the inequality

$$a(t) \leq a_0 < -n!\,\Delta^{-n}$$

is fulfilled, then all non-zero solutions of the equation

$$u^{(n)}(t) = a(t)u(t-\Delta)$$

are oscillatory at $+\infty$ At the same time, if a is non-positive and n is odd, the non-retarded equation (4) always has a non-oscillating solution.

The concepts of oscillation and non-oscillation on an interval are generally studied for linear homogeneous equations. They are of fundamental value in the theory of boundary value problems

Notes

13.6.1 Oscillating Solution

A solution $x(t)$ of a differential equation

$$x^{(n)} = f(t, x, x', ..., x^{(n-1)}), \qquad t \in [t_0, \infty),$$

with the property: There exists for any $t_1 \geq t_0$ a point $t_2 > t_1$ such that $x(t)$ changes sign on passing through it. In many applied problems there arises the question of the existence of an oscillating solution or the question whether all the solutions of equation (*) oscillate. Many sufficient conditions are known under which equation (*) has an oscillating solution . For example, any non-trivial solution of the equation $x'' + 2\delta x' + \omega^2 x = 0$ with constant coefficients is oscillating if $\delta^2 < \omega^2$; every non-trivial solution of the equation

$$x'' + p(t)x' + q(t)x = 0$$

with ω-periodic coefficients is oscillating if

$$\int_0^\infty dt \int_t^{t+\omega} q(s)\exp\left(-\int_s^t p(r)\,dr\right) ds \geq$$

$$\geq -\frac{1}{2}\left(1 - \exp\int_0^\omega p(t)\,dt\right)\int_0^\omega p(t)\,dt$$

and $q(t) \not\equiv 0$ on $[0, \omega]$.

In a number of applications there arises the question of the existence of oscillating solutions (in the above sense) of a system of ordinary differential equations. For example, in control theory one studies the oscillation relative to a given hyperplane $\sum_{i=1}^n c_i x_i = 0$ of the solutions $x(t) = (x_1(t), ..., x_n(t))$ of the system of equations $x' = f(t, x)$, that is, the question whether the function $\sigma(t) = \sum_{i=1}^n c_i x_i(t)$ oscillates. $[\alpha, \beta]$ -oscillating solutions are also studied; a bounded solution $x(t)$ of the system $x' = f(t, x)$ is called $[\alpha, \beta]$-oscillating if $\sigma(t)$ is oscillating and for any $t_1 \geq t_0$ there are points t_2 and t_3 such that $t_1 < t_2 < t_3$, $\sigma(t_2) < \alpha$, $\sigma(t_3) > \beta$, where $\alpha < 0 < \beta$. For the system $x' = f(x, t)$ there also exist other definitions of an oscillating solution.

13.7 LET'S SUM UP

- We study $x(t)$ and $y(t)$ are the nontrivial solutions, and a, b are two consecutive generalized zeros of $x(t)$, if
  $p_1(t) \geq p_2(t) > 0, \quad q_2(t) \geq q_1(t), \quad t \in [a, b] \cap \mathbf{T},$
  then $y(t)$ has at least one generalized zero on
- The Sturm–Liouville operator (cf. Sturm–Liouville equation)
  $(py')' + qy = 0, \; p > 0,$
- In this unit we learnt Oscillating Differential Equation and A non-zero solution of the equation
  $u^{(n)} = f(t, u, u', \ldots, u^{(n-1)}), \qquad n \geq 2,$
  where $f(t, 0, \ldots, 0) = 0$, is called oscillating at the point $+\infty$
- A second-order ordinary differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [\lambda w(x) - q(x)]\, y = 0,$$

13.8 KEYWORDS

Nontrivial Solution : A solution or example that is not trivial. Often, solutions or examples involving the number zero are considered trivial. Nonzero solutions or examples are considered nontrivial. For example, the equation x + 5y = 0 has the trivial solution (0, 0).

Riccati Equation : a Riccati equation in the narrowest sense is any first-order ordinary differential equation that is quadratic in the unknown function. In other words, it is an equation of the form

Oscillating : move or swing back and forth in a regular rhythm

13.9 QUESTIONS FOR REVIEW

Q. 1 State and Prove Sturm-Picone Comparison Theorem.

Q. 2 Define Sturm–Liouville operator .

Q. 3 State Sturm-Liouville Equation.

Q. 4 Write Picano Identity.

Q. 5 Define disconjugacy in Strum- Picano Comparison Theorem.

13.10 SUGGESTION READING AND REFERENCES

- W.A. Coppel, "Disconjugacy" , *Lecture Notes in Mathematics* , **220** , Springer (1971)
- U. Elias, "Oscillation theory of two-term differential equations" , Kluwer Acad. Publ. (1997)
- M. Gregus, "Third order linear differential equations" , Reidel (1987)
- Gustafson, G. B., "The nonequivalence of oscillation and nondisconjugacy" *Proc. Amer. Math. Soc.* , **25** (1970) pp. 254–260
- E. Hille, "Lectures on ordinary differential equations" , Addison-Wesley (1968)
- I.T. Kiguradze, T.A. Chanturia, "Asymptotic properties of solutions of nonautonomous ordinary differential equations" , Kluwer Acad. Publ. (1993) (In Russian)
- A.Yu. Levin, "Non-oscillation of solutions of the equation $x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x = 0$ " *Russian Math. Surveys* , **24** (1969) pp. 43–99 (In Russian)
- Z. Nehari, "The Schwarzian derivative and schlicht

functions" *Bull. Amer. Math. Soc.* , **55** (1949) pp. 545–551

- W.T. Reid, "Sturmian theory for ordinary differential equations" , Springer (1980)
- C.A. Swanson, "Comparison and oscillatory theory of linear differential equations" , Acad. Press (1968)
- W.F. Trench, "Canonical forms and principal systems for general disconjugate equation" *Trans. Amer. Math. Soc.* , **189** (1974) pp. 319–327

13.11 ANSWER TO CHECK IN PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 3

Q. 2 Check in Section 3

Check In progress-II

Answer Q. 1 Check in Section 4

Q. 2 Check in Section 5

UNIT 14: EIGENVALUE PROBLEMS

STRUCTURE

14.0 OBJECTIVES

- In this unit we study Eigen Value Problems with Reduction to a Discrete Problem
- We also study oscillation occurring in a dynamical system with its properties and examples.

- We study eigenvalues and eigenvectors technique with examples
- We study real eigen value and repeated eigenvalues
- **We study computation of eiegenvalues and computation of eiegenvectors**

14.1 INTRODUCTION

Methods for computing the eigen values and corresponding eigen functions of differential operators. Oscillations of a bounded elastic body are described by the equation

$$\frac{\partial^2 \phi}{\partial t^2} = L\phi,$$

where $L\phi$ is some differential expression. If one seeks solutions of (1) of the form

$$\phi = T(t)u(x),$$

the following equation for u is obtained:

$$L(u) + \lambda u = 0,$$

within a bounded domain, under certain homogeneous conditions on the boundary. The values of the parameter λ for which there exist non-zero solutions of (2) satisfying the homogeneous boundary conditions are called eigen values, and the corresponding solutions eigen functions. The resulting eigen value problem consists of determining the eigen values λ and the eigen functions corresponding to them.

The numerical solution of this problem proceeds in three steps:

1) reduction of the problem to a simpler one, for example, to an algebraic (discrete) problem.

2) specification of the precision of this discrete problem;

3) calculation of the eigen values for the discrete problem

14.2 REDUCTION TO A DISCRETE PROBLEM

The reduction of problem to a discrete model is usually carried out using the grid method and projection methods. It is natural to require that the basic properties of the original problem, such as the self-adjointness of an operator, be preserved in its discrete analogue.

One method of reduction is the integro-interpolation method. For example, consider the problem

$$\frac{d}{dx}\left(\frac{1}{p(x)}\frac{du}{dx}\right)+\lambda u=0, \qquad 0<x<1,$$
$$u(0)=0, \qquad u(1)=0\,.$$

This problem arises, for example, in the study of transverse vibrations of a non-homogeneous string and longitudinal vibrations of a non-homogeneous rod.

On the interval $[0,\,1]$, one introduces the uniform grid $\overline{\omega}$ with nodes $x_i=ih$, $i=0,\ldots,N$, $h=1/N$. To every node x_i, $i=1,\ldots,N-1$, is assigned an elementary domain $S_i=\{x:\ x_i-h/2\le x\le x_i+h/2\}$. Integration of equation (3) over S_i yields

$$\left.\begin{aligned} \frac{w_{i-1/2}-w_{i+1/2}}{h} &= \lambda\frac{1}{h}\int_{x_i-h/2}^{x_i+h/2} u\,dx, \\ w_{i\pm 1/2} &= \left(\frac{1}{p}\frac{du}{dx}\right)_{x=x_i\pm h/2}. \end{aligned}\right\}$$

Let

$$u=\ \text{const}\ =u_i, \qquad x_i-\frac{h}{2}\le x\le x_i+\frac{h}{2},$$
$$w=\ \text{const}\ =w_{i-1/2}, \qquad x_{i-1}\le x\le x_i\,.$$

Then

$$w_{i-1/2}=\frac{u_i-u_{i-1}}{h\alpha_i}, \qquad \alpha_i=\frac{1}{h}\int_{x_{i-1}}^{x_i} p(x)\,dx\,.$$

Substituting (6) into (5), one obtains

$$-\frac{1}{h}\left(\frac{v_{i+1}-v_i}{a_{i+1}h}-\frac{v_i-v_{i-1}}{a_i h}\right)=\lambda^h v_i, \qquad i=1,\ldots,N-1,$$

where v is the described grid function. The boundary conditions $v_0=0$, $v_N=0$ reduce to an algebraic eigen value problem:

$$Av=\lambda^h v,$$

where A is tri-diagonal symmetric matrix of order $n=N-1$.

The variational-difference method of reduction to a discrete problem is used when the eigen value problem can be formulated as a variational one. For example, the eigen values of the problem (3), (4) are the stationary values of the functional

$$\frac{D[u]}{H[u]}, \qquad D[u]=\int_0^1 \frac{1}{p}(x)\left(\frac{du}{dx}\right)^2 dx; \qquad H[u]=\int_0^1 u^2\,dx.$$

By changing the integrals to quadrature sums and the derivatives to difference relations, the discrete analogue of this functional has the form

$$\frac{D^h[v]}{H^h[v]}, \qquad D^h[v]=\sum_{i=1}^{N}\frac{1}{a_i}\left(\frac{v_i-v_{i-1}}{h}\right)^2 h;$$

$$H^h[v]=\sum_{i=0}^{N} v_i^2 h,$$

where a_i is the difference analogue of the coefficient $p(x)$, which can be calculated by formula (6). The discrete analogue of the problem (3), (4) is obtained from the necessary condition for an extremum:

$$\frac{\partial}{\partial v_i}\left(D^h[v]-\lambda^h H^h[v]\right)=0, \qquad i=1,\ldots,N-1.$$

Differentiation again leads to the problem (7) .

The projection-difference method of reduction to a discrete problem consists of the following. One chooses a linearly independent system of coordinate functions α_i, $i=1,\ldots,n$, and a linearly independent system of projection functions β_j, $j=1,\ldots,n$. One seeks approximate eigen functions in the form

$$\tilde{u} = \sum_{i=1}^{n} v_i \, \alpha_i \; .$$

The coefficients v_i and the approximate eigen values are determined by the condition

$$(L\tilde{u} + \widetilde{\lambda}\, \tilde{u}, \; \beta_j\,) = 0, \qquad j = 1\,, \ldots, n,$$

where $(,\,)$ is the scalar product in the Hilbert space. When the coordinate and projection systems coincide, one talks of the Bubnov–Galerkin method. If, in addition, the operator of the differential problem is self-adjoint, the method is called the Raleigh–Ritz method. In particular, for problem (3), (4), if all $\alpha_i = \beta_i$ satisfy (4), condition (8) takes the form

$$\sum_{i=1}^{n} v_i \int_0^1 \left(\frac{1}{p}(x) \frac{d\,\alpha_i}{dx} \frac{d\,\alpha_j}{dx} - \widetilde{\lambda}\;\alpha_i\;\alpha_j \right) dx = 0,$$

$$j = 1\,, \ldots, n\,.$$

In order to simplify the derivation of the algebraic problem, one chooses an almost-orthogonal system of functions α_i.

By taking as coordinate and projection systems functions of the form

$$\alpha_i\,(x) - \begin{cases} 0, & 0 \le x \le x_{i-1}, \\ \dfrac{x - x_{i-1}}{h}, & x_{i-1} \le x \le x_i, \\ \dfrac{x_{i+1} - x}{h}, & x_i \le x \le x_{i+1}, \\ 0, & x_{i+1} \le x \le 1, \end{cases}$$

where $x_i = ih$, it follows from (9) that

$$\frac{v_i - v_{i-1}}{a_i h} - \frac{v_{i+1} - v_i}{a_{i+1}\,h} +$$

$$- \lambda^h \;[(\,\rho\, v)_{i-1} + (\,\rho^*\;\; v)_i + (\,\rho\, v)_{i+1}] = 0,$$

where

$$a_i = \frac{1}{h} \int_{x_{i-1}}^{x_i} \frac{dx}{p}(x),$$

$$\rho^i = \int_{x_i}^{x_{i+1}} \alpha_i\;\alpha_{i+1}\;dx, \qquad \rho_i^* = \int_{x_{i-1}}^{x_{i+1}} \alpha_i^2\;dx\,.$$

Thus, together with the boundary conditions, one obtains a generalized eigen value problem:

$$Av = \lambda^h Dv, \qquad h = \frac{1}{N}.$$

Here A and D are tri-diagonal symmetric matrices of order $n = N - 1$.

These methods also yield discrete models of other equations. For example, for a rod:

$$\frac{d^2}{dx^2}\left(k\frac{d^2u}{dx^2}\right) = \lambda ru;$$

for a membrane:

$$\frac{\partial}{\partial x}\left(k_1\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_2\frac{\partial u}{\partial y}\right) + \lambda ru = 0;$$

and for a plate:

$$\frac{\partial^2}{\partial x^2}\left(k_{11}\frac{\partial^2 u}{\partial x^2} + k_{12}\frac{\partial^2 u}{\partial y^2}\right) + 2\frac{\partial^2}{\partial x \partial y}\left(k_{33}\frac{\partial^2 u}{\partial x \partial y}\right) +$$
$$+ \frac{\partial^2}{\partial y^2}\left(k_{21}\frac{\partial^2 u}{\partial x^2} + k_{22}\frac{\partial^2 u}{\partial y^2}\right) = \lambda ru.$$

The eigen vectors corresponding to λ_k^h satisfy the homogeneous system of algebraic equations:

$$(A - \lambda_k^h E)v_k = 0.$$

The problem of finding all eigen values and eigen vectors of a matrix A is called the complete problem of eigen values, and the problem of finding some eigen values of A is called the partial problem of eigen values.

In the case of algebraic systems corresponding to a given problem, the latter problem arises most frequently. The application of traditional solution methods requires a considerable amount of calculation, in view of the poor separation of the eigen values of A. In this case it is more effective to use modified gradient methods with spectrally equivalent operators and multi-grid methods

14.3 EIGEN OSCILLATION

Notes

An oscillation occurring in a dynamical system in the absence of an external action by perturbing it at the initial moment by an "external action" from a state of equilibrium. The nature of eigen oscillations is determined mainly by the internal forces determined by the physical structure of the system. The energy necessary for the movement enters the system from the "external action" at the initial moment of motion.

An example of eigen oscillations are the small oscillations of a conservative system with n degrees of freedom around a state of stable equilibrium. The equations of motion have the form

$$\sum_{i=1}^{n} (a_{si}\ddot{q}_i + c_{si}q_i) = 0, \qquad s = 1,\dots, n,$$

where the q_i are generalized coordinates and the a_{si}, c_{si} are constant coefficients. The general solution of (1) consists of the sum of n harmonic oscillations:

$$q_i = \sum_{j=1}^{n} A_j \, \Delta_i \, (k_j^2) \sin(k_j t + \beta_j), \qquad i = 1,\dots, n,$$

where A_j, β_j are constants of integration, k_j are eigen frequencies, i.e. roots of the frequency equation

$$\det \begin{pmatrix} c_{11} - a_{11}k^2 & \dots & c_{1n} - a_{1n}k^2 \\ \dots & \dots & \dots \\ c_{n1} - a_{n1}k^2 & \dots & c_{nn} - a_{nn}k^2 \end{pmatrix} = 0$$

(where it is assumed that there are no zero or multiple frequencies), and $\Delta_i \, (k_j^2)$ is the minor corresponding to the i-th column and last row of the determinant (2). The variables $A_j \, \Delta_i \, (k_j^2)$, $k_j t + \beta_j$ and β_j are the amplitude, phase and initial phase of the j-th harmonic, respectively. It follows from this example that harmonic oscillations of the same frequency for all coordinates arise in phase or contra-phase, and the distribution of amplitudes of oscillations of a given eigen frequency in the coordinates is determined by the physical structure of the system.

4.1 Check In Progress-I

Q. 1 Define The reduction of problem to a discrete model .

Solution : . . . .

Q.2 Define Eigen Oscillation.

Solution : . . . .

14.4 EIGENVALUES AND EIGENVECTORS TECHNIQUE

In this section we will discuss the problem of finding two linearly independent solutions for the homogeneous linear system

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

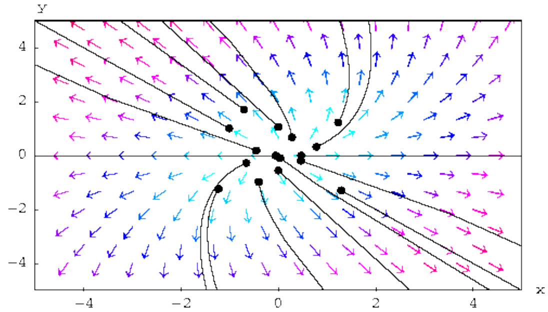
Let us first start with an example to illustrate the technique we will be developping.

Example: Draw the direction field of the linear system

Notes

$$\begin{cases} \dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = 3y \end{cases}$$

Answer: The following is the direction field:



Remark: From the above example we notice that some solutions lie on straight lines (can you spot them?). So it is natural to investigate whether and when an homogeneous linear system has solutions which are straight-lines.

14.4.1 Straight-Line Solutions

Consider the homogeneous linear system (in the matricial notation)

$$\frac{dY}{dt} = A\,Y$$

A straight-line solution is a vector function of the form

$$Y(t) = f(t)Y_0,$$

where Y_0 is a constant vector not equal to the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The vector Y_0 is the direction vector of the line on which the solution lives. Keep in mind that the solutions of the system may describe trajectories of moving objects. So, in this case, we may think of it as an object moving along a straight line.

Remark: Note that if $Y(t)$ is a straight-line solution, then $k\,Y(t)$ is also a straight-line solution.

Clearly, we have

$$\frac{dY}{dt} = f'(t)Y_0.$$

Therefore, we have

$$A\,Y = A\,f(t)\,Y_0 = f(t)\,AY_0 = f'(t)Y_0.$$

Since $A\,Y_0$ and Y_0 are constant vectors, we deduce that $\frac{f'(t)}{f(t)}$ is a constant function. Denote it by

$$\frac{f'(t)}{f(t)} = \lambda$$

Clearly, this is a first order differential equation which is linear as well as separable. Its solution is

$$f(t) = C\ \exp(\lambda\,t),$$

where C is an arbitrary constant. So, if a straight-line solution exists, it must be of the form

$$Y(t) = C\ \exp(\lambda\,t)\,Y_0,$$

where C is an arbitrary constant, and Y_0 is a non-zero constant vector which satisfies

$$A\,Y_0 = \lambda\,Y_0\ .$$

Note that we don't have to keep the constant C (read the above remark).

Let us illustrate the above ideas with an example.

Example: Find any straight-line solution to the system

$$\begin{cases} \dfrac{dx}{dt} & = & 2x - y \\ \dfrac{dy}{dt} & = & 3y \end{cases}$$

Answer: First, let us find the constant vector

$$Y_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

such that $A\,Y_0 = \lambda Y_0$ for some λ. Easy computations give

$$\begin{cases} 2x_0 & -y_0 & = & \lambda\, x_0 \\ & 3y_0 & = & \lambda\, y_0 \end{cases}$$

We have two cases:

Case 1. If $y_0 = 0$, then $x_0 \neq 0$ (since Y_0 is not the zero vector). The first equation gives $\lambda = 2$. In this case, the second equation forces $y_0 = 0$; therefore we have

$$Y_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We may ignore the constant x_0 (see the above remark). Therefore, the solution

$$Y(t) = \exp(2\,t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a straight-line solution to the system.

Case 2. If $y_0 \neq 0$, then from the second equation we get $\lambda = 3$. The first equation reduces to $2x_0 - y_0 = 3x_0$, or equivalently $y_0 = -x_0$. Therefore, we have

$$Y_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ -x_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We may again ignore the constant x_0. Hence, the solution

$$Y(t) = \exp(3\,t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a straight-line solution to the system.

So, we have found two straight-line solutions

$$\exp(2\,t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \exp(3\,t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Are these the only straight-lines? The answer is: "yes," but this will be discussed later.

Theorem: Straight-Line Solutions

Consider the homogeneous linear system

$$\frac{dY}{dt} = A\,Y$$

Any straight-line solution may be found in the form

$$Y(t) = \exp(\lambda\,t)\,Y_0,$$

where Y_0 ***is a non-zero constant vector which satisfies***

$$A\,Y_0 = \lambda\,Y_0\ .$$

The constant λ is called an **eigenvalue** of the matrix *A*, and Y_0 is called an **eigenvector** associated to the eigenvalue λ of the matrix *A*. Clearly, if Y_0 is an eigenvector associated to λ, then $k\,Y_0$ is also an eigenvector associated to λ. Our next target is to find out how to search for the eigenvalues and eigenvectors of a matrix.

14.4.2 Computation of Eiegenvalues

Consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and assume that λ is an eigenvalue of *A*. Then there must exist a non-zero vector $Y_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, such that $A\,Y_0 = \lambda\,Y_0$. This equation may be rewritten as the algebraic system

$$\begin{cases} a\,x_0 + b\,y_0 & = & \lambda\,x_0 \\ c\,x_0 + d\,y_0 & = & \lambda\,y_0 \end{cases}$$

which is equivalent to the system

$$\begin{cases} (a-\lambda)\,x_0 & + & b\,y_0 & = & 0 \\ c\,x_0 & + & (d-\lambda)\,y_0 & = & 0 \end{cases}$$

Since both x_0 and y_0 can not be equal to zero at the same time, we must have the determinant of the system equal to zero. That is,

$$det\begin{pmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

,

which reduces to the algebraic equation

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

Note that the above equation is independent of the vector Y_0. This equation is called the **Characteristic Polynomial** of the system.

Example: Find the characteristic polynomial and the eigenvalues of the matrix

$$\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$$

Answer: The characteristic polynomial is given by

$$det \begin{pmatrix} (2-\lambda) & -1 \\ 0 & (3-\lambda) \end{pmatrix} = \lambda^2 - 5\lambda + 6 = 0.$$

This is a quadratic equation. Its only roots are $\lambda = 2$ and $\lambda = 3$. These are the eigenvalues of the matrix.

14.4.3 Computation of Eiegen vectors

Assume λ is an eigenvalue of the matrix *A*. An eigenvector Y_0 associated to λ is given by the matricial equation

$$A\, Y_0 = \lambda\, Y_{0.}$$

Set $Y_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. Then, the above matricial equation reduces to the algebraic system

$$\begin{cases} a\, x_0 + b\, y_0 & = & \lambda\, x_0 \\ c\, x_0 + d\, y_0 & = & \lambda\, y_0 \end{cases}$$

which is equivalent to the system

$$\begin{cases} (a-\lambda)\, x_0 & + & b\, y_0 & = & 0 \\ c\, x_0 & + & (d-\lambda)\, y_0 & = & 0 \end{cases}$$

Notes

Since λ is known, this is now a system of two equations and two unknowns. You must keep in mind that if Y_0 is an eigenvector, then $k\,Y_0$ is also an eigenvector.

Example: Consider the matrix

$$\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}.$$

Find all the eigenvectors associated to the eigenvalue $\lambda = 3$.

Answer: In the above example we checked that in fact $\lambda = 3$ is an eigenvalue of the given matrix. Let Y_0 be an eigenvector associated to the eigenvalue $\lambda = 3$. Set $Y_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. Then we must have

$$\begin{cases} (2-3)\,x_0 & + & -\,y_0 & = & 0 \\ 0 & + & (3-3)\,y_0 & = & 0 \end{cases}$$

which reduces to the only equation

$$-x_0 - y_0 = 0,$$

which yields $y_0 = -x_0$. Therefore, we have

$$Y_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ -x_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Note that we have all of the eigenvectors associated to the eigenvalue $\lambda = 3$.

Conclusion

In order to find the straight-line solution to the homogeneous linear system

$$\frac{dY}{dt} = A\,Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} Y$$, perform the following steps:

*First, we look for the eigenvalues through the characteristic polynomial

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

This is a quadratic equation which has one double real root, or two distinct real roots, or two complex roots.

*Once an eigenvalue λ is found from the characteristic polynomial, then we look for the eigenvectors Y_0 associated to it through the matricial equation

$$A\,Y_0 = \lambda\,Y_0.$$

If you find a parameter factorized in front of Y_0, there will be no need to keep it;

*For an eigenvalue λ and an associated eigenvector Y_0, a straight-line solution will be given by

$$Y(t) = \exp(\lambda\, t)\,Y_0.$$

Remark: It is not hard to show that two straight-line solutions generated by two different eigenvalues are in fact linearly independent. Combined with the results of the previous section we now see how straight-lines may be used to help find the solutions of an homogeneous linear system. This technique is also related to the case of second order differential equation with constant coefficients. Indeed, consider the second order differential equation

$$a\,y'' + b\,y' + c\,y = 0.$$

Notes

Set $v = y'$. Then the second order differential equation is equivalent to the first order system

$$\begin{cases} \dfrac{dy}{dt} = v \\ \dfrac{dv}{dt} = -\dfrac{c}{a}\, y - \dfrac{b}{a}\, v \end{cases}$$

The matrix coefficient of the system is

$$\begin{pmatrix} 0 & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} \end{pmatrix}.$$

The characteristic polynomial is

$$\lambda^2 - \left(0 - \frac{b}{a}\right)\lambda + \frac{c}{a} = 0,$$

which is equivalent to the equation

$$a\,\lambda^2 + b\,\lambda + c = 0.$$

We recognize the characteristic equation associated to the second order differential equation.

Check In Progress-II

Q. 1 Define Straight Line Solution.

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

Q.2 Find the characteristic polynomial and the eigenvalues of the matrix

$$\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$$

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

14.4.4 Real Eigenvalues

Consider the linear homogeneous system

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\[2ex] \dfrac{dy}{dt} = cx + dy \end{cases}$$

In order to find the eigenvalues, consider the characteristic polynomial

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

In this section we will consider the case of the quadratic equation above when it has two distinct real roots (that is, if $(a+d)^2 - 4(ad-bc) > 0$). The roots (eigenvalues) are

$$\lambda_1 = \frac{(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}}{2},$$

and

$$\lambda_2 = \frac{(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

Here we know that the differential system has two linearly independent straight-line solutions

Notes

$$\exp(\lambda_1\ t)\ V_1 \quad \text{and} \quad \exp(\lambda_2\ t)\ V_2,$$

where V_1 (respectively V_2) is an eigenvector associated to the eigenvalue λ_1 (respectively λ_2). We also know that the general solution (which describes all of the solutions) to the system has the form

$$Y(t) = k_1\ \exp(\lambda_1\ t)\ V_1 + k_2\ \exp(\lambda_2\ t)\ V_2.$$

Keep in mind that V_1 and V_2 are two constant vectors.

Let us discuss the behavior of the solutions when $t \longrightarrow +\infty$ (meaning the future) and when $t \longrightarrow -\infty$ (meaning the past). Since the eigenvalues are distinct, one is bigger than the other one. Assume that we have

$$\lambda_1 < \lambda_2.$$

It is easy to see that we have

$$\begin{aligned} Y(t) &= \exp(\lambda_1\ t)\left(k_1\ V_1 + k_2\ \exp\left((\lambda_2 - \lambda_1)\ t\right)\ V_2\right) \\ &= \exp(\lambda_2\ t)\left(k_1\ \exp\left((\lambda_1 - \lambda_2)\ t\right)\ V_1 + k_2\ V_2\right) \end{aligned}$$

Behavior when $t \longrightarrow +\infty$

In this case we will consider the equation

$$Y(t) = \exp(\lambda_2\ t)\left(k_1\ \exp\left((\lambda_1 - \lambda_2)\ t\right)\ V_1 + k_2\ V_2\right).$$

Since

$$\lim_{t \to +\infty} \exp\left((\lambda_1 - \lambda_2)\ t\right) = 0,$$

(because $\lambda_1 - \lambda_2 < 0$) then it is clear that when $t \longrightarrow +\infty$, we have

$$Y(t) \approx \exp(\lambda_2\ t)\ k_2\ V_2.$$

Behavior when $t \to -\infty$

In this case we will consider the equation

$$Y(t) = \exp(\lambda_1\ t)\ \left(k_1\ V_1 + k_2\ \exp\left((\lambda_2 - \lambda_1)\ t\right)\ V_2\right)$$

Since

$$\lim_{t \to -\infty} \exp\left((\lambda_2 - \lambda_1)\ t\right) = 0$$

(because $\lambda_2 - \lambda_1 > 0$) then it is clear that when $t \to -\infty$, we have

$$Y(t) \approx \exp(\lambda_1\ t)\ k_1\ V_1$$

Remark: Since the two eigenvalues are real numbers, we have three cases to consider depending on their signs:

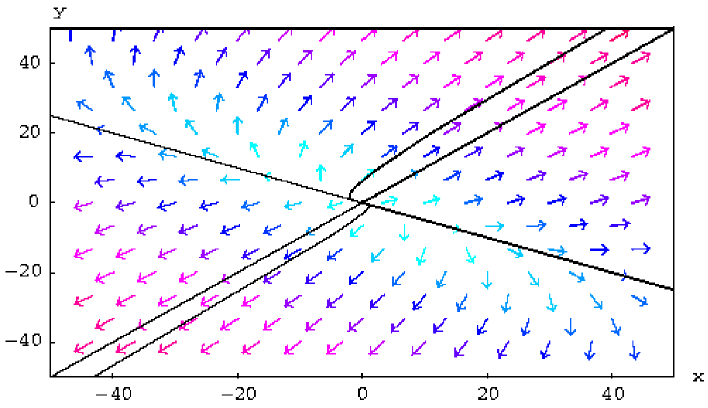
Case 1: Both are positive

$$0 < \lambda_1 < \lambda_2.$$

In this case we have

$$\lim_{t \to -\infty} Y(t) = 0,$$

meaning that the solutions emanate from the origin (if you go to the past, you will die at the origin). When $t \to +\infty$, $Y(t)$ explodes.



In this case the origin plays the role of a **source**. Clearly, the origin is the only equilibrium point.

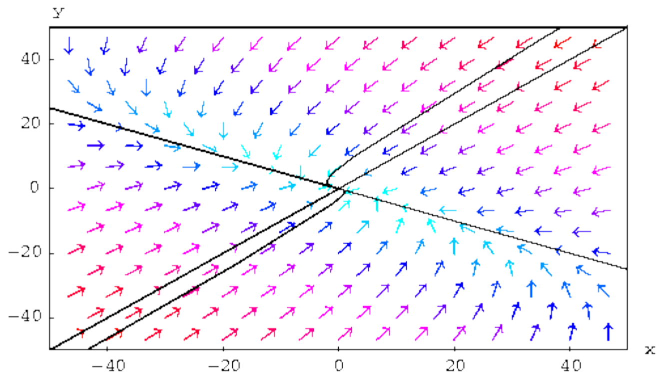
Case 2: Both eigenvalues are negative

$$\lambda_1 < \lambda_2 < 0.$$

In this case we have

$$\lim_{t \to +\infty} Y(t) = 0,$$

meaning that in the future the solutions die at the origin. When $t \to -\infty$, $Y(t)$ explodes.

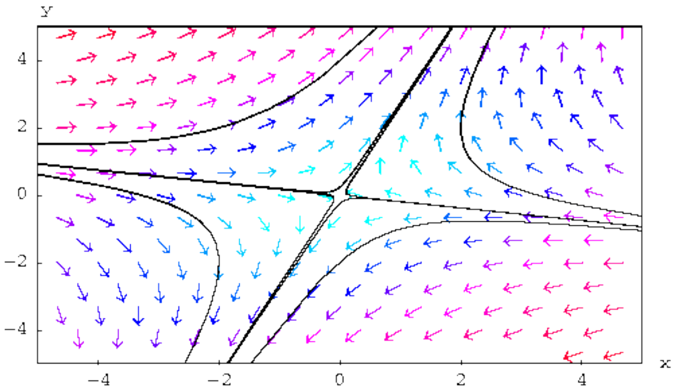


In this case, the origin plays the role of a **sink**. Clearly, the origin is the only equilibrium point.

Case 3: The eigenvalues have different signs

$$\lambda_1 < 0 < \lambda_2.$$

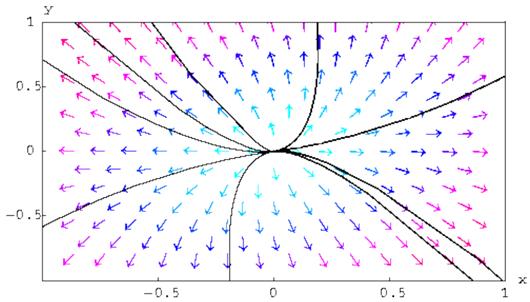
In this case, the origin behaves like a **saddle**.



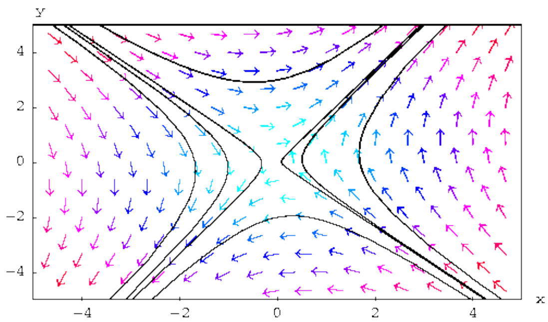
Remark: It is clear from the above discussions that one may decide about the signs of the eigenvalues just by looking at some solutions on the phase plane (depending whether we have a saddle, a sink or a source).

Example: Consider the three phase planes and decide about the sign-distribution of the associated eigenvalues.

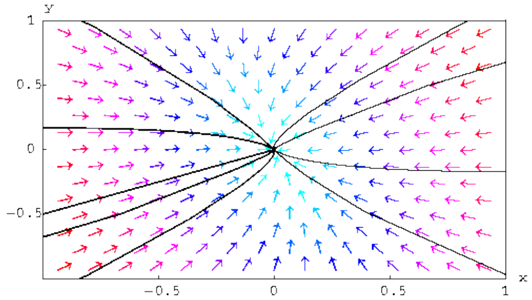
Phase Plane I



Phase Plane II



Phase Plane III



Answer: For the phase-plane I, the origin is a source. Therefore, the two eigenvalues are both positive.

For the phase-plane II, the origin is a saddle. Hence, the two eigenvalues are opposite signs.

For the phase-plane III, the origin is a sink. Hence, the two eigenvalues are negative.

Example: Consider the harmonic oscillator equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 0.$$

Discuss the behavior of the spring-mass.

Answer: First, translate this equation to the system

$$\frac{dY}{dt} = A\,Y,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -5 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y \\ v \end{pmatrix} \text{ where } v = \frac{dy}{dt}.$$

The characteristic polynomial of this system is

$$\lambda^2 + 5\lambda + 1 = 0.$$

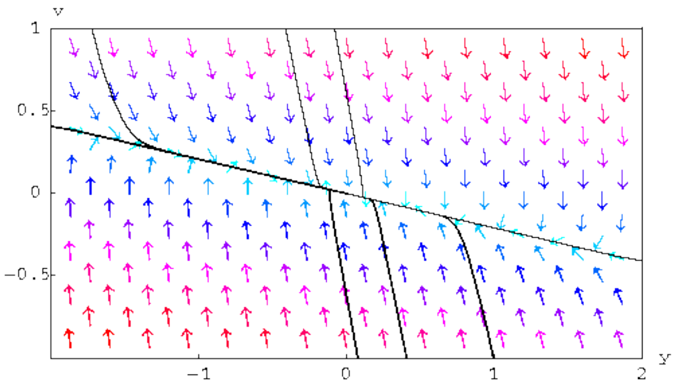
The eigenvalues are

$$\lambda = \frac{-5 \pm \sqrt{21}}{2}.$$

It is clear that both of them are negative. Hence, the origin is a sink. Meaning that, regardless of the initial condition, the mass will always tend to its equilibrium, or rest, position.

Note that if V is an eigenvector associated to the biggest eigenvalue $(-5+\sqrt{21})/2$, then all the solutions tend to the origin tangent to that vector V. In this case we have

$$V = \begin{pmatrix} 1 \\ \dfrac{-5+\sqrt{21}}{2} \end{pmatrix}.$$



Remark: The case when one of the two eigenvalues is zero will be discussed in another section separately.

Example 1 : Consider the harmonic oscillator with spring constant $k_s = 4$, damping constant $k_d = 5$, and the mass m=1.

(1) Write down the second order equation governing this physical system. Use the letter y for the spring's displacement from its rest position.

(2) Convert this equation into a linear system of first order differential equations.

(3) Solve the system.

(4) Find the particular solution which satisfies the initial conditions

$$y(0) = 0 \text{ and } y'(0) = 1$$

(5) Discuss the long-term behavior of the system. Is this conclusion probable?

Answer:(1) The differential equation is

$$my'' + k_d y' + k_s y = 0.$$

Using the values for the constants, we get

$$y'' + 5y' + 4y = 0.$$

(2) Set *y*'=*v*, then we have

$$v' = y'' = -5y' - 4y = -4y - 5v.$$

Hence, we have the system

$$\begin{cases} y' &= v \\ v' &= -4y - 5v; \end{cases}$$

(3) In order to solve the above system, we first need to find the eigenvalues of the system. Note that the matrix coefficient is

$$\begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}.$$

The characteristic equation is given by

$$\lambda^2 + 5\lambda + 4 = 0.$$

Its roots are

$$\lambda = \frac{-5 \pm \sqrt{25 - 16}}{2} = \frac{-5 \pm 3}{2},$$

which gives

$$\lambda = -1 \quad \text{or} \quad \lambda = -4\ .$$

For every eigenvalue, we need to find an eigenvector.

- $\lambda = -1$. Let *V* be an associated eigenvector such that

$$V = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The vector *V* must satisfy the system of algebraic equations

$$\begin{cases} y_0 = -x_0 \\ -4x_0 - 5y_0 = -y_0 \end{cases}$$

Clearly, the two equations reduce to the same equation

$$x_0 + y_0 = 0.$$

Hence, we have

$$V = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We choose

$$V = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- $\lambda = -4$. Let *V* be an associated eigenvector such that,

$$V = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The vector *V* must satisfy the system of algebraic equations

$$\begin{cases} y_0 = -4x_0 \\ -4x_0 - 5y_0 = -4y_0 \end{cases}$$

Clearly, the two equations reduce to the same equation

$$4x_0 + y_0 = 0.$$

Hence, we have

$$V = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

We choose

$$V = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Now we are ready to write down the general solution

$$Y = k_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix},$$

where

$$Y = \begin{pmatrix} y \\ v \end{pmatrix} .$$

(4) In order to find the solution to the harmonic oscillator system which satisfies the initial conditions $y(0)= 0$ and $y'(0)=1$, we need the general solution which gives y. From the general solution to the system we get

$$y = k_1 e^{-t} + k_2 e^{-4t},$$

and $v = -k_1 e^{-t} - 4k_2 e^{-4t}$. The equation giving v is obvious and can be obtained from y since $v=y'$ (you may want to check that we did not make any mistakes). The initial conditions imply

$$\begin{cases} k_1 + k_2 & = 0 \\ -k_1 - 4k_2 & = 1 \end{cases}$$

Solving it we get

$$k_1 = \frac{1}{3} \text{ and } k_2 = -\frac{1}{3}.$$

Therefore, the solution is

$$y = \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} .$$

(5) The long-term behavior of the solution is now obvious since

$$\lim_{t \to +\infty} y(t) = 0,$$

meaning that the system tends to its rest position. Note that since the eigenvalues are both negative, it was clear from the outset that the solution will tend to its unique equilibrium position.

Example 2 : Consider a harmonic oscillator for which the differential equation is

$$y'' + 4y' + 2y = 0,$$

and suppose that mass *m*=1, the damping constant $k_d = 4$, and the spring constant $k_s = 2$. Rewrite this equation as a linear system of differential equations. Solve it, then find the particular solution which satisfies the initial conditions

$$y(0) = 1 \quad y'(0) = 2 .$$

Answer. Set *v*=*y*'. Then we have

$$v' = y'' = -4y' - 2y = -2y - 4v .$$

This gives us the system

$$\begin{cases} \dfrac{dy}{dt} = v \\ \dfrac{dv}{dt} = -2y - 4v \end{cases}$$

which in matrix form may be rewritten as

$$\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -4 \end{pmatrix} Y$$

where

$$Y = \begin{pmatrix} y \\ v \end{pmatrix}$$

In order to solve this system, we need the characteristic equation

$$\begin{vmatrix} 0-\lambda & 1 \\ -2 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 2 = 0$$

Its roots are given by the quadratic formulas

$$\lambda = \frac{-4 \pm \sqrt{16-8}}{2} = -2 \pm \sqrt{2}\,.$$

Note that you have to be very careful here since any mistake at finding correctly the roots will generate a far bigger mistakes and waist of time!! Next we need to find the associated eigenvectors.

Case $\lambda = -2 + \sqrt{2}$. Denote by $V_1(x_0, y_0)$ the associated eigenvector. The system giving V_1 is

$$\begin{cases} y_0 = (-2+\sqrt{2})x_0 \\ -2x_0 - 4y_0 = (-2+\sqrt{2})y_0 \end{cases}$$

Since

$$\frac{-2}{(-2+\sqrt{2})+4} = -2+\sqrt{2}$$

(which you should check as an exercise), then the two equations are identical. Hence we take $y_0 = (-2+\sqrt{2})x_0$. If we choose $x_0 = 1$, we get

$$V_1 = \begin{pmatrix} 1 \\ -2+\sqrt{2} \end{pmatrix}$$

Case $\lambda = -2 - \sqrt{2}$. Similar calculations give the associated eigenvector

$$V_2 = \begin{pmatrix} 1 \\ -2-\sqrt{2} \end{pmatrix}$$

Therefore the general solution is given by

$$Y = k_1 e^{(-2+\sqrt{2})t} \begin{pmatrix} 1 \\ -2+\sqrt{2} \end{pmatrix} + k_2 e^{(-2-\sqrt{2})t} \begin{pmatrix} 1 \\ -2-\sqrt{2} \end{pmatrix}$$

where k_1 and k_2 are two parameters.

From the above equation giving *Y*, we may find the solution *y* to our second differential equation as

$$y = k_1 e^{(-2+\sqrt{2})t} + k_2 e^{(-2-\sqrt{2})t}$$

We are almost done except that we need to find the specific solution which satisfies the initial condition

$$y(0) = 1 \quad y'(0) = 2$$

These two conditions imply

$$\begin{cases} k_1 + k_2 = 1 \\ (-2+\sqrt{2})k_1 + (-2-\sqrt{2})k_2 = 2 \end{cases}$$

The second equation gives

$$-2(k_1 + k_2) + \sqrt{2}(k_1 - k_2) = 2$$

since $k_1 + k_2 = 1$, we get $\sqrt{2}(k_1 - k_2) = 4$ which implies $k_1 - k_2 = \frac{4}{\sqrt{2}}$. Hence we have

$$\begin{cases} k_1 + k_2 = 1 \\ k_1 - k_2 = \frac{4}{\sqrt{2}} \end{cases}$$

which implies

$$k_1 = \frac{1}{2}\left(1 + \frac{4}{\sqrt{2}}\right)$$

and

$$k_1 = \frac{1}{2}\left(1 - \frac{4}{\sqrt{2}}\right)$$

which yields

$$y = \frac{1}{2}\left(1 + \frac{4}{\sqrt{2}}\right) e^{(-2+\sqrt{2})t} + \frac{1}{2}\left(1 - \frac{4}{\sqrt{2}}\right) e^{(2-\sqrt{2})t} .$$

Example 3 : Consider the linear system

$$\begin{cases} \frac{dx}{dt} = 2x + y \\ \frac{dy}{dt} = -y. \end{cases}$$

Find the matrix coefficient of the system. Then, discuss the fate of the long term behavior of the solutions. If they go to infinity, discuss how.

Answer: The matrix coefficient of the system is

$$\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Note that if you have the wrong matrix coefficient the conclusion about the solutions may totally differ from the right answer!

In order to find the general solution we need the characteristic equation

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0 .$$

Its roots are given by the quadratic formulas

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2},$$

which gives $\lambda = 2$ or $\lambda = -1$. Next, we need to find the associated eigenvectors.

Case $\lambda = 2$. Denote by $V_1(x_0, y_0)$ the associated eigenvector. The system giving V_1 is

$$\begin{cases} 2x_0 + y_0 &= 2x_0 \\ -y_0 &= 2y_0. \end{cases}$$

The two equations lead to the same equation $y_0 = 0$. If we choose $x_0 = 1$, we get

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Case $\lambda = -1$. Denote by $V_2(x_0, y_0)$, the associated eigenvector. The system giving V_2 is

$$\begin{cases} 2x_0 + y_0 &= -x_0 \\ -y_0 &= -y_0 \end{cases}$$

The second equation is worthless and the first one implies $y_0 = -3x_0$. If we choose $x_0 = 1$, we get

$$V_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Therefore, the general solution is given by

$$Y = k_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 1 \\ -3 \end{pmatrix},$$

where k_1 and k_2 are two parameters.

We know that since the system has one positive eigenvalue the solutions will tend to infinity as *t* goes to $+\infty$. We also know that the solutions will get closer and closer to the straight-line solution which corresponds to the biggest eigenvalues. In this case, the line generated by the vector

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

is clearly the x-axis.

Check In Progress-III

Problem : Consider the harmonic oscillator with spring constant $k_s = 4$, damping constant $k_d = 5$, and the mass *m*=1.

Q. 1 Convert Above equation into a linear system of first order differential equations.

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

Q.2 Find the particular solution which satisfies the initial conditions

$$y(0) = 0 \ \text{ and } \ y'(0) = 1$$

Solution : . . . .
. . . .
. . . .
. . . .

14.4.5 Repeated Eigenvalues

Consider the linear homogeneous system

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\[2ex] \dfrac{dy}{dt} = cx + dy \end{cases}$$

In order to find the eigenvalues consider the Characteristic polynomial

$$\lambda^2 - (a + d)\lambda + ad - bc = 0$$

In this section, we consider the case when the above quadratic equation has double real root (that is if $(a+d)^2-4(ad-bc)=0$) the double root (eigenvalue) is

$$\lambda_1=\frac{(a+d)}{2}$$

In this case, we know that the differential system has the straight-line solution

$$\exp(\lambda_1\ t)\ V_1$$

where V_1 is an eigenvector associated to the eigenvalue λ_1. We also know that the general solution (which describes all the solutions) of the system will be

$$Y(t)=k_1\ \exp(\lambda_1\ t)\ V_1+k_2\ Y_2$$

where Y_2 is another solution of the system which is linearly independent from the straight-line solution $\exp(\lambda_1\ t)\ V_1$. Therefore, the problem in this case is to find Y_2.

Search for a second solution.

Let us use the vector notation. The system will be written as

$$\frac{dY}{dt}=AY$$

where *A* is the matrix coefficient of the system. Write

$$Y_1=\exp(\lambda_1\ t)\ V_1$$

The idea behind finding a second solution Y_2, linearly independent from Y_1, is to look for it as

$$Y_2=\exp(\lambda_1\ t)\ \left(tV_1+V_2\right)$$

where V_2 is some vector yet to be found. Since

$$\frac{dY_2}{dt}=\lambda_1 e^{\lambda_1 t}tV_1+e^{\lambda_1 t}V_1+\lambda_1 e^{\lambda_1 t}V_2$$

and

$$AY_2=\lambda_1 e^{\lambda_1 t}tV_1+e^{\lambda_1 t}AV_2$$

(where we used $AV_1=\lambda_1 V_1$), then (because Y_2 is a solution of the system) we must have

$$\lambda_1 e^{\lambda_1 t} t V_1 + e^{\lambda_1 t} V_1 + \lambda_1 e^{\lambda_1 t} V_2 = \lambda_1 e^{\lambda_1 t} t V_1 + e^{\lambda_1 t} A V_2 .$$

Simplifying, we obtain

$$e^{\lambda_1 t}\left(V_1 + \lambda_1 V_2\right) = e^{\lambda_1 t}\left(AV_2\right)$$

or

$$V_1 + \lambda_1 V_2 = AV_2$$

This equation will help us find the vector V_2. Note that the vector V_2 will automatically be linearly independent from V_1 (why?). This will help establish the linear independence of Y_2 from Y_1.

Example. Find two linearly independent solutions to the linear system

$$\begin{cases} \dfrac{dx}{dt} = 2x + y \\ \\ \dfrac{dy}{dt} = 2y \end{cases}$$

Answer. The matrix coefficient of the system is

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

In order to find the eigenvalues consider the Characteristic polynomial

$$\lambda^2 - 4\lambda + 4 = 0$$

Since $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, we have a repeated eigenvalue equal to 2. Let us find the associated eigenvector V_1. Set

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then we must have $AV_1 = 2V_1$ which translates into

$$\begin{cases} 2x + y = 2x \\ 2y = 2y \end{cases}$$

This reduces to *y*=0. Hence we may take

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Next we look for the second vector V_2. The equation giving this vector is $AV_2 = \lambda_1 V_2 + V_1$ which translates into the algebraic system

$$\begin{cases} 2x + y = 2x + 1 \\ 2y = 2y \end{cases}$$

where

$$V_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

Clearly we have y=1 and x may be chosen to be any number. So we take x=0 for example to get

$$V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore the two independent solutions are

$$Y_1 = e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } Y_2 = e^{2t} \left\{ t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

The general solution will then be

$$Y = c_1 \; e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \; e^{2t} \left\{ t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

14.5 QUALITATIVE ANALYSIS OF SYSTEMS WITH REPEATED EIGENVALUES

Recall that the general solution in this case has the form

$$Y(t) = k_1 \; \exp(\lambda \, t) \, V_1 + k_2 \; \exp(\lambda \, t) \left(tV_1 + V_2 \right)$$

where λ is the double eigenvalue and V_1 is the associated eigenvector. Let us focus on the behavior of the solutions when $t \to +\infty$ (meaning the future). We have two cases

- If $\lambda < 0$, then clearly we have

$$Y(t) \to (0, 0) \text{ when } t \to +\infty$$

In this case, the equilibrium point (0,0) is a sink. On the other hand, when t is large, we have

$$Y(t) \approx te^{\lambda t} k_2 \; V_1$$

So the solutions tend to the equilibrium point tangent to the straight-line solution. Note that is $k_2 = 0$, then the solution is the straight-line solution which still tends to the equilibrium point.

Example 4 . Consider the system

$$\begin{cases} \dfrac{dx}{dt} &= 2x + y \\ \dfrac{dy}{dt} &= -x + 4y \end{cases}$$

1.Find the general solution.

2.Find the solution which satisfies the initial condition

$$Y_0 = (1, -1)$$

3.Draw some solutions in the phase-plane including the solution found in 2.

Answer. The matrix coefficient of the system is

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

In order to find the eigenvalues consider the characteristic polynomial

$$\lambda^2 - 6\lambda + 9 = 0$$

Since $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$, we have a repeated eigenvalue equal to 3. Let us find the associated eigenvector V_1. Set

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then we must have $AV_1 = 3V_1$ which translates into

$$\begin{cases} 2x + y = 3x \\ -x + 4y = 3y \end{cases}$$

This reduces to *y*=*x*. Hence we may take

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Next we look for the second vector V_2. The equation giving this vector is $AV_2 = \lambda_1 V_2 + V_1$ which translates into the algebraic system

$$\begin{cases} 2x + y = 3x + 1 \\ -x + 4y = 3y + 1 \end{cases}$$

where

$$V_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

Clearly the two equations reduce to the equation $y - x = 1$ or $y = 1 + x$, where x may be chosen to be any number. So if we take $x=0$ for example, we get

$$V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore the two independent solutions are

$$Y_1 = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } Y_2 = e^{3t} \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

The general solution will then be

$$Y = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

In order to find the solution which satisfies the initial condition

$$Y_0 = (1, -1)$$

we must have

$$Y(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This implies $c_1 = 1$ and $c_2 = -2$. Hence the solution is

$$Y = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2 e^{3t} \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = e^{3t} \begin{pmatrix} 1 - 2t \\ -1 - 2t \end{pmatrix}$$

14.5.1 Systems with Zero as an Eigenvalue

We discussed the case of system with two distinct real eigenvalues, repeated (nonzero) eigenvalue, and complex eigenvalues. But we did not discuss the case when one of the eigenvalues is zero. In fact, it is easy to see that this happen if and only if we have more than one equilibrium point (which is (0,0)). In this case, we will have a line of equilibrium points (the direction vector for this line is the eigenvector associated to the eigenvalue zero).

Example. Find the general solution to

$$\begin{cases} \dfrac{dx}{dt} &= 2x - y \\[2ex] \dfrac{dy}{dt} &= -2x + y \end{cases}$$

Answer. The characteristic polynomial of this system is

$$\lambda^2 - (2+1)\lambda + 0 = 0$$

which reduces to $\lambda^2 - 3\lambda = 0$. The eigenvalues are $\lambda = 0$ and $\lambda = 3$. Let us find the associated eigenvectors.

- For $\lambda = 0$, set

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

The equation $AV = 0\ V$ translates into

$$\begin{cases} 2x - y & = & 0 \\ -2x + y & = & 0 \end{cases}$$

The two equations are the same. So we have $y = 2x$. Hence an eigenvector is

$$V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- For $\lambda = 3$, set

$$V_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

The equation $AV = 3\ V$ translates into

$$\begin{cases} 2x - y & = & 3x \\ -2x + y & = & 3y \end{cases}$$

The two equations are the same (as $-x-y=0$). So we have $y = -x$. Hence an eigenvector is

$$V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore the general solution is

$$Y = k_1 e^{0t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Note that all the solutions are line parallel to the vector $V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. When $t \to +\infty$, the trajectory goes to infinity. But when $t \to -\infty$, the trajectory converge to the equilibrium point on the line of equilibrium points (that is passing by (0,0) and having V_1 as a direction vector).

The general case is very similar to this example. Indeed, assume that a system has 0 and $\lambda \neq 0$ as eigenvalues. Hence if V_1 is an eigenvector associated to 0 and V_2 an eigenvector associated to λ, then the general solution is

$$Y = k_1 e^{0t} V_1 + k_2 e^{\lambda t} V_2 = k_1 V_1 + k_2 e^{\lambda t} V_2$$

We have two cases, whether $\lambda > 0$ or $\lambda < 0$.

- If $k_2 = 0$, then $Y(t) = k_1 V$ is an equilibrium point.
- If $K_2 \neq 0$, then the solution is a line parallel to the vector V_2.

Moreover, we have when $t \to +\infty$

- if $\lambda > 0$, the solution tends away from the line of equilibrium;
- if $\lambda < 0$, the solution tends to the equilibrium point $k_1 V_1$ along a line parallel to V_2.

14.5.2 Complex Eigenvalues

Consider the linear homogeneous system

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

The Characteristic polynomial is

$$\lambda^2 - (a + d)\lambda + ad - bc = 0$$

In this section, we consider the case when the above quadratic equation has complex roots (that is if $(a + d)^2 - 4(ad - bc) < 0$). The roots (eigenvalues) are

$$\frac{(a + d) \pm i\sqrt{4(ad - bc) - (a + d)^2}}{2} = \alpha \pm i\beta$$

where

$$\alpha = \frac{(a + d)}{2} \text{ and } \beta = \frac{\sqrt{4(ad - bc) - (a + d)^2}}{2}$$

In this case, the difficulty lies with the definition of

$$e^{(\alpha + i\beta)t}$$

In order to get around this difficulty we use **Euler's formula**

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Therefore, we have

$$e^{(\alpha + i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}\left(\cos\left(\beta t\right) + i\sin\left(\beta t\right)\right)$$

In this case, the eigenvector associated to $\lambda = \alpha + i\beta$ will have complex components.

Example. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

Answer. The characteristic polynomial is

$$\lambda^2 + \lambda + 1 = 0$$

Its roots are

$$\frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Set $\lambda = \frac{-1 + i\sqrt{3}}{2}$. The associated eigenvector V is given by the equation $AV = \lambda V$. Set

$$V = \begin{pmatrix} x \\ y \end{pmatrix}$$

The equation $AV = \lambda V$ translates into

$$\begin{cases} y & = & \lambda x \\ -x - y & = & \lambda y \end{cases}$$

Since $\lambda = -\frac{1}{1 + \lambda}$, then the two equations are the same (which should have been expected, do you see why?). Hence we have $y = \lambda x$ which implies that an eigenvector is

$$V = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-1 + i\sqrt{3}}{2} \end{pmatrix}$$

We leave it to the reader to show that for the eigenvalue $\frac{-1-i\sqrt{3}}{2}$, the eigenvector is

$$V = \begin{pmatrix} 1 \\ \frac{-1-i\sqrt{3}}{2} \end{pmatrix}$$

Let us go back to the system

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$$

with complex eigenvalues $\alpha \pm i\beta$. Note that if V, where

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

is an eigenvector associated to $\alpha + i\beta$, then the vector

$$\bar{V} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix},$$

(where $\bar{v}$ is the conjugate of v) is an eigenvector associated to $\alpha - i\beta$.

On the other hand, we have seen that

$$e^{(\alpha+i\beta)t}V \quad \text{and} \quad e^{(\alpha-i\beta)t}\bar{V}$$

are solutions. Note that these solutions are complex functions. In order to find real solutions, we used the above remarks. Set

$$v_1 = x_1 + iy_1 \quad \text{and} \quad v_2 = x_2 + iy_2$$

then we have

$$e^{(\alpha+i\beta)t}\, v_1 = e^{\alpha t}\Big(\cos(\beta t) + i\sin(\beta t)\Big)\Big(x_1 + iy_1\Big)$$

which gives

$$e^{(\alpha+i\beta)t}\, v_1 = e^{\alpha t}\Big\{\Big(x_1\cos(\beta t) - y_1\sin(\beta t)\Big) + i\Big(y_1\cos(\beta t) + x_1\sin(\beta t)\Big)\Big\}$$

Similarly we have

$$e^{(\alpha+i\beta)t}\, v_2 = e^{\alpha t}\Big\{\Big(x_2\cos(\beta t) - y_2\sin(\beta t)\Big) + i\Big(y_2\cos(\beta t) + x_2\sin(\beta t)\Big)\Big\}$$

Putting everything together we get

$$e^{(\alpha+i\beta)t}V = e^{\alpha t}\begin{pmatrix} \big(x_1\cos(\beta t) - y_1\sin(\beta t)\big) + i\big(y_1\cos(\beta t) + x_1\sin(\beta t)\big) \\ \big(x_2\cos(\beta t) - y_2\sin(\beta t)\big) + i\big(y_2\cos(\beta t) + x_2\sin(\beta t)\big) \end{pmatrix}$$

Clearly this implies $e^{(\alpha+i\beta)t}V = Y_1 + iY_2$ where

$$Y_1 = e^{\alpha t}\begin{pmatrix} x_1\cos(\beta t) - y_1\sin(\beta t) \\ x_2\cos(\beta t) - y_2\sin(\beta t) \end{pmatrix} \text{ and } Y_2 = e^{\alpha t}\begin{pmatrix} y_1\cos(\beta t) + x_1\sin(\\ y_2\cos(\beta t) + x_2\sin(\end{pmatrix}$$

It is easy to see that we have

$$e^{(\alpha-i\beta)t}\bar{V} = Y_1 - iY_2$$

Since the sum and difference of solutions lead to another solution, then both Y_1 and Y_2 are solutions of the system. These are real solutions. It is very easy to check in fact that they are linearly independent. Let us summarize the above technique.

Summary (of the complex case). Consider the system

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

- Write down the characteristic polynomial

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

and find its roots

$$\lambda = \frac{(a+d) \pm i\sqrt{4(ad-bc) - (a+d)^2}}{2} = \alpha \pm i\beta$$

we are assuming that $(a+d)^2 - 4(ad-bc) < 0$. Note that at this step, you need to know α and β. The common mistake is to forget to divide by 2.

- Find an eigenvector V associated to the eigenvalue $\alpha + i\beta$. Write down the eigenvector as

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

- Two linearly independent solutions are given by the formulas

$$Y_1 = e^{\alpha t} \begin{pmatrix} x_1 \cos(\beta t) - y_1 \sin(\beta t) \\ x_2 \cos(\beta t) - y_2 \sin(\beta t) \end{pmatrix} \text{ and } Y_2 = e^{\alpha t} \begin{pmatrix} y_1 \cos(\beta t) + x_1 \sin(\beta t) \\ y_2 \cos(\beta t) + x_2 \sin(\beta t) \end{pmatrix}$$

- The general solution is

$$Y = k_1 Y_1 + k_2 Y_2$$

where k_1 and k_2 are arbitrary numbers. Note that in this case, we have

$$Y = e^{\alpha t} \left\{ k_1 \begin{pmatrix} x_1 \cos(\beta t) - y_1 \sin(\beta t) \\ x_2 \cos(\beta t) - y_2 \sin(\beta t) \end{pmatrix} + k_2 \begin{pmatrix} y_1 \cos(\beta t) + x_1 \sin(\beta t) \\ y_2 \cos(\beta t) + x_2 \sin(\beta t) \end{pmatrix} \right\}$$

Example. Consider the harmonic oscillator

$$y'' + y' + y = 0$$

Find the general solution using the system technique.

Answer. First we rewrite the second order equation into the system

$$\begin{cases} \dfrac{dy}{dt} = v \\ \\ \dfrac{dv}{dt} = -y - v \end{cases}$$

The matrix coefficient of this system is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

We have already found the eigenvalues and eigenvectors of this matrix. Indeed the eigenvalues are

$$\lambda = \frac{-1 \pm i\sqrt{3}}{2}$$

Hence we have

$$\alpha = -\frac{1}{2} \text{ and } \beta = \frac{\sqrt{3}}{2}$$

The eigenvector associated to $\lambda = \alpha + i\beta$ is

$$V = \begin{pmatrix} 1 \\ -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2} \end{pmatrix}$$

Next we write down the two linearly independent solutions

$$Y_1 = e^{-1/2t} \begin{pmatrix} \cos\left(\dfrac{\sqrt{3}}{2}t\right) \\ -\dfrac{1}{2}\cos\left(\dfrac{\sqrt{3}}{2}t\right) - \dfrac{\sqrt{3}}{2}\sin\left(\dfrac{\sqrt{3}}{2}t\right) \end{pmatrix}$$

and

$$Y_2 = e^{-1/2t}\begin{pmatrix} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$$

The general solution of the equivalent system is

$$Y = k_1 Y_1 + k_2 Y_2$$

or

$$Y = e^{-1/2t}\left\{ k_1 \begin{pmatrix} \cos\left(\frac{\sqrt{3}}{2}t\right) \\ -\frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} + k_2 \begin{pmatrix} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} \right\}$$

Below we draw some solutions.

Since we are looking for the general solution of the differential equation, we only consider the first component. Therefore we have

$$y = k_1 e^{-1/2t}\cos\left(\frac{\sqrt{3}}{2}t\right) + k_2 e^{-1/2t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

You may want to check that the second component is just the derivative of *y*.

14.5.3 Qualitative Analysis of Systems with Complex Eigenvalues

Recall that in this case, the general solution is given by

$$Y = e^{\alpha t}\left\{ k_1 \begin{pmatrix} x_1\cos(\beta t) - y_1\sin(\beta t) \\ x_2\cos(\beta t) - y_2\sin(\beta t) \end{pmatrix} + k_2 \begin{pmatrix} y_1\cos(\beta t) + x_1\sin(\beta t) \\ y_2\cos(\beta t) + x_2\sin(\beta t) \end{pmatrix} \right\}$$

The behavior of the solutions in the phase plane depends on the real part α. Indeed, we have three cases:

- **the case:** $\alpha < 0$**.** The solutions tend to the origin (when $t \to +\infty$) while spiraling. In this case, the equilibrium point is called a **spiral sink**.

- **The case:** $\alpha > 0$ The solutions explode or get away from the origin (when $t \to +\infty$) while spiraling. In this case, the equilibrium point is called a **spiral source**.

- **The case:** $\alpha = 0$ The solutions are periodic. This means that the trajectories are closed curves or cycles. In this case, the equilibrium point is called a **center**.

14.6 LET'S SUM UP

In this unit we learnt reduction of the integro-interpolation method. For example, consider the problem

$$\frac{d}{dx}\left(\frac{1}{p(x)}\frac{du}{dx}\right) + \lambda u = 0, \qquad 0 < x < 1,$$
$$u(0) = 0, \qquad u(1) = 0\,.$$

We learnt A straight-line solution is a vector function of the form

$$Y(t) = f(t)Y_0,$$

We learnt the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and assume that λ is an eigenvalue of *A*.

Then there must exist a non-zero vector $Y_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, such that $A\,Y_0 = \lambda\,Y_0$.

We also studied Complex Eigen Components

$$e^{(\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t}\left(\cos\left(\beta t\right) + i\sin\left(\beta t\right)\right)$$

In this case, the eigenvector associated to $\lambda = \alpha + i\beta$ will have complex

components.

14.7 KEYWORD

Notes

EigenValues : each of a set of values of a parameter for which a differential equation has a non-zero solution (an eigenfunction) under given conditions.

Eigen Vector : a nonzero vector that is mapped by a given linear transformation of a vector space onto a vector that is the product of a scalar multiplied by the original vector. — called also characteristic vector

Tri-diagonal : tridiagonal (not comparable) (linear algebra, of a matrix) Having nonzero elements only in the main diagonal and the diagonals directly above and below it.

Discrete Model : Discrete modelling is the discrete analogue of continuous modelling. In discrete modelling, formulae are fit to discrete data—data that could potentially take on only a countable set of values, such as the integers, and which are not infinitely divisible.

14.8 QUESTIONS FOR REVIEW

Q. 1 Find any straight-line solution to the system

$$\begin{cases} \dfrac{dx}{dt} = 2x - y \\[2ex] \dfrac{dy}{dt} = 3y \end{cases}$$

Q. 2 Consider the matrix

$$\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}.$$

Find all the eigenvectors associated to the eigenvalue $\lambda = 3$.

Q. 3 Find the general solution to

$$\begin{cases} \dfrac{dx}{dt} = 2x - y \\[2ex] \dfrac{dy}{dt} = -2x + y \end{cases}$$

Q. 4 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

Q. 5 Consider the harmonic oscillator

$$y'' + y' + y = 0$$

Find the general solution using the system technique

Q. 6 Consider the matrix

$$\begin{bmatrix} 1 & 5 \\ -2 & -1 \end{bmatrix}$$

Find all the eigenvectors associated to the eigenvalue $\lambda = 3$.

14.9 SUGGESTION READING AND REFERENCES

[1] S. Gould, "Variational methods for eigenvalue problems" , Univ. Toronto Press (1957)

[2] L. Collatz, "Eigenwertaufgaben mit technischen Anwendungen" , Akad. Verlagsgesell. Geest u. Portig K.-D. (1949)

[3] D.K. Faddeev, V.N. Faddeeva, "Computational methods of linear algebra" , Freeman (1963) (Translated from Russian)

[4] V.V. Voevodin, "Numerical methods of algebra" , Moscow (1966) (In Russian)

[5] J.H. Wilkinson, "The algebraic eigenvalue problem" , Oxford Univ. Press (1969)

[6] B.N. Parlett, "The symmetric eigenvalue problem" , Prentice-Hall (1980)

[7] V.K. Saul'ev, "On estimating the error in finding eigenfunctions by the finite difference method" *Vychisl. Mat.* , **1** (1957) pp. 87–115 (In Russian)

[8] A.N. Tikhonov, A.A. Samarskii, "The Sturm–Liouville difference problem" *USSR Comput. Math. Math. Phys.* , **1** : 4 (1961) pp. 939–961 *Zh. Vychisl. Mat. i Mat. Fiz.* , **1** : 5 (1961) pp. 784–805

[9] A.A. Samarskii, "Theorie der Differenzverfahren" , Akad. Verlagsgesell. Geest u. Portig K.-D. (1984) (Translated from

Russian)

[10] Shou Hao, "The Sturm–Liouville difference problem for a fourth order equation with discontinuous coefficients" *USSR Comput. Math. Math. Phys.* , **3** : 6 (1963) pp. 1383–1406 *Zh. Vychisl. Mat. i Mat. Fiz.* , **3** : 6 (1963) pp. 1014–1031

[11] V.G. Prikazchikov, "High-accuracy homogeneous difference schemes for the Sturm–Liouville problem" *USSR Comput. Math. Math. Phys.* , **9** : 2 pp. 76–106 *Zh. Vychisl. Mat. Mat. Fiz.* , **9** : 2 (1969) pp. 315–336

[12] V.G. Prikazchikov, "A difference eigenvalue problem for a second-order operator with mixed boundary conditions" *USSR Comput. Math. Math. Phys.* , **22** : 3 (1982) pp. 165–172 *Zh. Vychisl. Mat. Mat. Fiz.* , **22** : 3 (1982) pp. 655–662

14.10 ANSWER TO CHECK IN PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 3

Q. 2 Check in Section 4

Check In progress-II

Answer Q. 1 Check in Section 5.1

Q. 2 Check in Section 5.2

Check In progress-III

Answer Q. 1 Check in Section 5.5

Q. 2 Check in Section 5.5